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Partitioning A Graph In Alliances And Its Application To Data Clustering

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by

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Abstract

Any reasonably large group of individuals, families, states, and parties exhibits the phenomenon of subgroup formations within the group such that the members of each group have a strong connection or bonding between each other. The reasons of the formation of these subgroups that we call alliances differ in different situations, such as, kinship and friendship (in the case of individuals), common economic interests (for both individuals and states), common political interests, and geographical proximity. This structure of alliances is not only prevalent in social networks, but it is also an important characteristic of similarity networks of natural and unnatural objects. (A similarity network defines the links between two objects based on their similarities). Discovery of such structure in a data set is called clustering or unsupervised learning and the ability to do it automatically is desirable for many applications in the areas of pattern recognition, computer vision, artificial intelligence, behavioral and social sciences, life sciences, earth sciences, medicine, and information theory.

In this dissertation, we study a graph theoretical model of alliances where an alliance of the vertices of a graph is a set of vertices in the graph, such that every vertex in the set is adjacent to equal or more vertices inside the set than the vertices outside it. We study the problem of partitioning a graph into alliances and identify classes of graphs that have such a partition. We present results on the relationship between the existence of such a

partition and other well known graph parameters, such as connectivity, subgraph structure, and degrees of vertices. We also present results on the computational complexity of finding such a partition.

An alliance cover set is a set of vertices in a graph that contains at least one vertex from every alliance of the graph. The complement of an alliance cover set is an alliance free set, that is, a set that does not contain any alliance as a subset. We study the properties of these sets and present tight bounds on their cardinalities. In addition, we also characterize the graphs that can be partitioned into alliance free and alliance cover sets.

Finally, we present an approximate algorithm to discover alliances in a given graph. At each step, the algorithm finds a partition of the vertices into two alliances such that the alliances are strongest among all such partitions. The strength of an alliance is defined as a real number p, such that every vertex in the alliance has at least p times more neighbors in the set than its total number of neighbors in the graph). We evaluate the performance of the proposed algorithm on standard data sets.

Dedicated to my daughter *Aleesha*

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CHAPTER 1

INTRODUCTION

1.1 Motivation

The word 'Alliance' means a bond or connection between individuals, families, states, or parties. In the real world, alliances are found in many varieties, each having different properties, for example

- alliances of nations for mutual support in war (to attack against a common enemy or to defend against an aggressor), in economy, or for other common interests,
- alliances of different political parties,
- alliances of people who unite by relationship or friendship,
- alliances of companies with common economic interests.

Inspired by the alliances between nations at war, alliances in graphs were first introduced by Hedetniemi, et. al.[HHK00]. Assuming that nations are represented by vertices in a graph and edges correspond to possible relations (of either friendship or hostility) between nations, they defined an alliance to be a set of vertices in a graph such that each vertex is adjacent to at least as many vertices inside the set (including itself) as outside it. In other words, every nation in an alliance has at least as many friends in the alliance as it has enemies outside the alliance. One can think of a vertex in an alliance being able to defend itself or any of its neighboring allies (by strength of numbers) from possible attack by vertices outside the alliance. That is why such an alliance is called a defensive alliance. Within the similar context of national security, more types of alliances were defined in the latter studies, which include offensive alliances[FFG02], powerful alliances[BDH02], secure alliances[BDH04], and alliances in directed graphs[Lan04](The definitions and other properties of each of these alliances are presented in Chapter 2). In realistic scenarios, the amount of support or hostility is not determined by the strength of numbers, but by the economic power of the nation, the size and effectiveness of its forces, geographical conditions, etc. These factors can be modelled by using edge weighted graphs. In this case, the weight of an edge between two vertices represents the amount of support or hostility between the two corresponding nations. A defensive alliance in an edge weighted graph is a set of vertices of the graph such that for each vertex in the alliance, the sum of the weights of its edges within the alliance is at least as large as the sum of the weights of its edges outside the alliance.

In general, this concept of alliances can be applied whenever a grouping of objects, with respect to some common property, is the matter of concern. We may assume that the vertices in a graph are objects that we seek to group and the edges define the common property the objects share (say, similarity of objects with each other). Then by using the above definition of weighted defensive alliance, an alliance is a set of objects such that the similarity of objects within the alliance is more than the similarity outside it. Such grouping or *clustering* of objects by their similarities with each other (and/or dissimilarities with respect to other groups) is one of the fundamental properties of living organisms [Sok77]. A human being at a very early stage of life would doubtlessly recognize and differentiate between many clusters of objects such as clusters of people vs trees, clusters of birds vs fish, clusters of men vs women, etc. These clusters can be seen as a part of unsupervised learning in human beings that allows them to infer important characteristics and patterns from the given input stimuli. Thus, clustering is a higher level intellectual activity necessary to our understanding of nature and modelling of human intellect and perceptual processes.

The problem for automatically finding clusters of similar object (data) arises in many areas of studies such as pattern recognition, computer vision, artificial intelligence, behavioral and social sciences, life sciences, earth sciences, medicine, and information theory [And73]. This automatic clustering of objects (data) is by no means a trivial task, which is evidenced by the overwhelming amount of existing literature focussing on this problem in almost every field of science. A large repertoire of mathematical techniques [DH, Eve93a, Har75, Mir96] is used including graph theoretical models and vertex partitioning schemes, such as connected component, clique, graph coloring, min-cut, minimum spanning tree, and minimum normalized cut.

Despite this interest and effort, the clustering problem in general is far from solved. Proposed methods are largely ad hoc and/or specialized to specific problems. One particular difficulty in finding such a solution is the formalization of the notions of clusters and clustering processes [FP03]. It is clear that what we should be doing is forming clusters that are helpful to a particular application, but this criterion has not been formalized in any useful way.

Using this as our motivation, we study different types of alliances in graphs. Of particular interest are the problems of partitioning the vertex set of a graph into different types of alliances. A number of interesting problems in graph theory and algorithm design arise from the study. We study the associated parameters, their properties, inter-relation and the extremal cases. Computational complexity and algorithms of the resulting problems are also investigated.

In particular, we identify classes of graphs that have partitions into defensive alliances and strong defensive alliances based on their connectivity, and subgraph properties. We also characterize special classes of graphs, such as, regular graphs and line graphs, that have these partitions. We characterize the graphs that have partitions into strong defensive alliance free sets and strong defensive alliance cover sets (An alliance cover set is a set of vertices in a graph that contains at least one vertex from every alliance of the graph. An alliance free set is a set that does not contain any alliance as a subset). In addition, we prove tight bounds on the sizes of strong defensive alliance, defensive alliance free sets, and defensive alliance cover sets.

We also present an approximate algorithm for data clustering. The algorithm clusters the data by splitting large insufficiently similar clusters into smaller clusters by finding a partition of the vertices into two alliances such that the alliances are strongest among all such partitions. The strength of an alliance is defined as a real number p, such that every vertex in the alliance has at least p times more neighbors in the set than its total number of neighbors in the graph). We applied this algorithm for different clustering applications and tested it on standard data sets.

1.2 Definitions and Notation

In the remainder of this text, we will assume the following notation.

Consider a graph $G = (V, E)$ without loops or multiple edges, having *vertex set* V and *edge set* E. If $|V| = n$ and $|E| = m$, we say that G is of *order* n and *size* m. For any vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u : uv \in E\}$, while the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The degree of a vertex v is defined as $\deg(v) = |N(v)|$. For a set S and vertex v, we denote $\deg_S(v) = |N(v) \cap S| = |N_S(v)| = \deg(v) - \deg_{V-S}(v)$. Similarly, $N[v] \cap S = N_S[v]$. The open and closed neighborhoods of sets of vertices $S \subset V$ are defined as follows: $N(S) = \bigcup_{v \in S} N(v)$, and $N[S] = N(S) \cup S$. The *boundary* of a set S is the set $\partial S = \bigcup_{v \in S} N(v) - S$. A graph $G' = (V', E')$ is a *subgraph* of a graph $G = (V, E)$, written $G' \subseteq G$ if $V' \subseteq V$ and $E' \subseteq E$. If $S \subset V$ is a subset of the vertex set, the subgraph induced by S is the graph $G[S] = (S, E \cap (S \times S))$.

An edge *cutset* of a connected graph G is a set $S \subseteq E(G)$ such that $G - S$ is disconnected. If no proper subset of S is a cutset, then S is called *minimal cutset.* If S has the minimum number of edges among all cutsets then S is called a *minimum cutset* of G. Let V_1 and V_2 partition V. The set of edges of the cutset S which have one end vertex in V_1 and the other in V_2 is denoted as $\langle V_1, V_2 \rangle$. The same notation will be used for the vertex partition formed by V_1 and V_2. The meaning of notation will be obvious by the context within which it is used. *Edge connectivity* $\kappa_1(G)$ of a graph G is the minimum number of edges whose removal from G results in a disconnected graph. Similarly, *Vertex connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal from G results in a disconnected graph or the trivial graph.

A set $K \subseteq V$ is called a *vertex cover* of graph G if every edge of G has at least one end vertex in K. A vertex cover K of G is *minimum* if G has no vertex cover K' with $|K'| < |K|$. The number of vertices in a minimum vertex cover of G is called the *vertex covering number* of G and is denoted by α_0.

An *independent set* of graph G is a subset S of V such that no two vertices of S are adjacent in G. An independent set S of G is *maximum* if G has no independent set S' with $|S'| > |S|$. The number of vertices in a maximum independent set of G is called the *independence number* or *stability number* of G and is denoted by $\beta_0(G)$.

A set of vertices D in a graph G is a dominating set in G if every vertex not in D is adjacent to a vertex in D. The minimum cardinality of a dominating set of G is the *domination number* $\gamma(G)$.

Other terminology and notation will be introduced as needed. In general, we follow that in [Wes01].

1.3 Dissertation Outline

The dissertation is organized as follows: In Chapter 2, different types of alliances in graphs are introduced, and their properties, associated parameters and the computational complexities are discussed. In Chapter 3, the problem of finding a bipartition of a graph into defensive alliances (Satisfactory partitioning problem) is studied, where the conditions for the existence and computability of such partitions and the computational complexities of the related problems are presented. The concept of alliance-free sets and alliance-cover sets is introduced in Chapter 4. In Chapter 5, we characterize the graphs whose vertex set can be partition into alliance-free (cover) sets.

CHAPTER 2

ALLIANCES IN GRAPHS

2.1 Introduction

In order to study the properties of real world alliances, the graph theoretical definition of alliance was first introduced by Hedetniemi, et. al.[HHK00]. Though they formalized the notion based on the alliances formed by different nations (to defend each other or attack a common enemy), the concept can be generalized to other situations where a grouping of similar elements is a matter of concern. In this chapter, we will present different types of alliances and their variants along with the associated parameters and problems of interest.

We begin with the definition of defensive alliance. Consider a graph $G = (V, E)$ without loops or multiple edges. A non-empty set of vertices $S \subseteq V$ is called a *defensive alliance* if and only if for every $v \in S$, $|N[v] \cap S| \geq |N(v) \cap (V - S)|$. Using national security issues to illustrate these concepts, one can think of a vertex in an alliance S being able to *defend* itself or any of its neighboring allies from possible attack by vertices in $V - S$. Since each vertex in a defensive alliance S has at least as many vertices from its closed neighborhood in

S as it has in $V - S$, by strength of numbers, we say that every vertex in S can be defended from possible attack by vertices in $V - S$. A defensive alliance is called *strong* if for every vertex $v \in S, |N[v] \cap S| > |N(v) \cap (V - S)|$, i.e., $\deg_S(v) \geq \deg_{V-S}(v)$. In this case, we say that every vertex in S is *strongly defended*.

Though the notion of alliances in graphs was first introduced and formally defined in [HHK00], similar concepts had been the topic of several studies in the past. The bipartition of the vertex set of a graph in degree constraint sets can be traced back to the problem of *unfriendly partition* of graphs introduced by Borodin and Kostochka [BK] in 1977. A partition is said to be unfriendly if each vertex has as many or more neighbors outside the set in which it occurs than inside it. The problem has been studied by Bernardi [Ber87], Cowan and Emerson [CE], Aharoni, Milner and Prikry [AMP90] and Shelah and Milner [SM90].

In [GK01], Gerber and Kobler studied a similar but complementary problem where the bipartition of vertex set was sought such that each vertex has as many or more neighbors inside the set in which it occurs than outside it. Such a partition is called *Satisfactory Partition* and was also the focus of study in [SD02a], where necessary and sufficient conditions for graphs to have such a partition were presented. In terms of alliances, a satisfactory partition is basically a bipartition of vertex sets in strong defensive alliances. In [SD02a], the term *cohesive sets* was used for the strong defensive alliances.

Another similar concept is that of web communities [FLG00, Bri02]. The emergence of the world wide web, enormous increase in computing power, data storage and communication speed in recent years has lead to the availability of huge amounts of data. The task of indexing and categorizing such data is difficult. One way of categorizing the Web is to divide it into communities each of which would be rich in content specific to a topic. Flake et al [FLG00] define *web community* as a set of sites that have more links (in either direction) to the members of the community than to non-members.

The concept of alliance is also related to signed [DHH95b] and minus [DHH99] dominating functions in graphs. A function $f : V \rightarrow \{-1, +1\}$ is called a *signed dominating function* if for every vertex $v \in V$, $\sum_{w \in N[v]} f(w) \geq 1$. It is easy to see that if $\langle V_{-1}, V_1 \rangle$ is a partition defined by f^{-1}, V_1 is a strong defensive alliance. Similarly, a function $g : V \rightarrow \{-1, 0, +1\}$ is called a *minus dominating function* if for every vertex $v \in V$, $\sum_{w \in N[v]} g(w) \geq 1$. Once again, V_1 is a strong defensive alliance if $\langle V_{-1}, V_0, V_1 \rangle$ is a partition defined by g^{-1}. Signed and minus domination in graphs are also studied in [DHH96, DGH96, Fav94, HHS94, HHS95, Zel96].

A set $S \subseteq V$ is called *nearly perfect* [DHH95a] if for all $v \in V - S$, $deg_S(v) \leq 1$. Similarly, an *efficient dominating set* [BHJ93] is a set such that $\forall v \in V - S$, $deg_S(v) = 1$. A *2-packing* is a set $S \subseteq V$ if $\forall v \in V$, $deg_S[v] \leq 1$. From these definitions, it is easy to see that the complements of every nearly perfect set, efficient dominating set, and 2-packing are defensive alliances.

A set $S \subseteq V$ is called an $\alpha-$ *dominating set* [DHL00], for some α, $0 < \alpha \leq 1$, if for every vertex $v \in V - S$, $deg_S(v) \geq \alpha \deg(v)$. Thus, if $\alpha \leq 1/2$, the complement of an $\alpha-$dominating set is a strong defensive alliance.

2.2 Types of Alliances

Other than defensive alliances defined in the previous section, several other types of alliances were proposed in [HHK00], while other generalizations have also been presented recently. In this section, we review some of these.

A concept similar to defensive alliances is that of *offensive alliance*, where a non empty set of vertices $S \subseteq V$ is called an *offensive alliance* if and only if for every $v \in \partial S$, $|N(v) \cap S| \geq |N[v] \cap (V - S)|$. Here, we say that every vertex in ∂S is vulnerable to possible attack by vertices in S (by strength of numbers). An offensive alliance is called a *strong offensive alliance* if for ever vertex $v \in \partial S$, $|N(v) \cap S| > |N[v] \cap (V - S)|$.

In [SD03], the concepts of defensive and offensive alliances were generalized to defensive(offensive) k-alliances, where the strength of an alliance is related to the value of parameter k. A vertex v in set $S \subseteq V$ is said to be $k-$satisfied with respect to S if $\deg_S(v) \geq \deg_{V-S}(v) + k$. A set S is a *defensive k-alliance* if all vertices in S are $k-$satisfied with respect to S, where $-\Delta < k \leq \Delta$. Note that a defensive $(-1)-$alliance is a "defensive alliance" (as defined in [HHK00]), and a defensive $0-$alliance is a "strong defensive alliance" or "cohesive set" [SD02a]. Similarly, a set $S \subseteq V$ is an offensive $k-$alliance if $\forall v \in \partial S, \deg_S(v) \geq \deg_{V-S}(v) + k$, where $-\Delta + 2 < k \leq \Delta$. Here, an offensive $1-$alliance is an "offensive alliance" and an offensive $2-$alliance is a "strong offensive alliance" (as defined in [FFG02, HHK00]).

Another obvious generalization is that of defensive(offensive) p-alliances, where instead of forcing a vertex to have a fixed difference between its allies and enemy vertices regardless of the total number of allies, we restrict a vertex to have p times more neighbors in its alliance than its total number of neighbors in the graph, where p is any real number such that $0 \leq p \leq 1$. Formally, a set S is a *defensive p-alliance* if for all vertices $v \in S, \deg_S(v) \geq p \deg_{V-S}(v)$. Similarly, a set $S \subseteq V$ is an offensive p-alliance if $\forall v \in \partial S, \deg_S(v) \geq p \deg_{V-S}(v)$. Once again, there is a significant overlap between the concept of p-alliances and that of α-dominating sets [DHL00].

An alliance is called a *powerful alliance* [BDH02] if it is both defensive and offensive. This concept can be expressed by the single condition that for every vertex $v \in N[S]$, $|N[v] \cap S| \geq |N[v] - S|$. Since a powerful alliance S is defensive, it can defend every vertex in S from possible attack by the vertices in ∂S, and since it is offensive, it can effectively attack every vertex in ∂S. Furthermore, a powerful alliance can also defend every vertex in ∂S from attack by vertices in $N[\partial S] - N[S]$, i.e., S can defend itself and all its neighbors.

All alliances above involve the defense of a single vertex. In more realistic settings, alliances are formed so that any attack on the entire alliance or any subset of the alliance can be forestalled. A defensive alliance S is called *secure* [BDH04] if, for any subset $X \subset S$, an attack on all the vertices of X can be repelled. Formally, for any $S \subseteq V$ and $X = \{x_1, x_2, \ldots, x_k\} \subseteq S$, an *attack* of X is any k disjoint sets $A = \{A_1, A_2, \ldots, A_k\}$ for which $A_i \subseteq N[x_i] - S$, $1 \leq i \leq k$. A *defense* of X is any k disjoint sets $D = \{D_1, D_2, \ldots, D_k\}$ for which $D_i \subseteq N[x_i] \cap S$, $1 \leq i \leq k$. Defense D of X is said to defend against attack A, with respect to the set S, whenever $|D_i| \geq |A_i|$ for $1 \leq i \leq k$. Alternatively, X is defendable from attack by A. The set X is *S-secure* if it is defendable from attack by A. When $X = S$ and S is $S - secure$, S is said to be *secure*.

An alliance (of any type) is called *global* [HHH02] if it affects every vertex in $V - S$, i.e., every vertex in $V - S$ is adjacent to at least one member of the alliance S. In other words an alliance S is global if it is also a dominating set.

Note that all these alliances can be easily generalized to edge weighted and/or vertex weighted graphs. Let $f : E \rightarrow \Re$ and $g : V \rightarrow \Re$. A set $S \subseteq V$ is called *weighted* defensive alliance, if for all $v \in S$, $\sum_{u \in N_S[v]} f(u,v)g(u) \geq \sum_{u \in N_{V-S}(v)} f(u,v)g(u)$. Alliances defined earlier can be generalized to weighted graphs in a similar fashion. For the un-weighted cases, the functions f and g may both be assumed to be equal to 1.

2.3 Alliance Numbers

In this section, we will introduce some parameters associated with the different types of alliances. In general we will refer to all types of alliances simply as alliances and the parameters are collectively called alliance numbers. An alliance (of some type) is called *critical* or *minimal* if no proper subset of S is an alliance (of the same type). In the rest of this text we will ignore the parenthesized phrases emphasizing that the alliances of same types are the topic of concern and will assume that it will always be the case unless specified otherwise.

Note that the property of being an alliance is not necessarily hereditary, i.e., a set contained in an alliance is not necessarily an alliance. We define an alliance S to be *locally minimal* or *locally critical*, if for all $v \in S$, $S - \{v\}$ is not an alliance. Generalizing, we define an alliance S to be $r-$critical or $r-$minimal if for all $T \subset S$ such that $|T| = r$, $S - T$ is not an alliance. An alliance is minimum if it is a minimal alliance of smallest cardinality.

Similarly, an alliance S is maximal if it is not a proper subset of any other alliance. It is $k-$maximal if for all $T \subseteq V - S$, such that $|T| = k$, $S \cup T$ is not an alliance. An alliance is maximum if it is a maximal alliance of maximum cardinality.

The cardinality of minimum alliance of a graph G is called the *alliance number* of G, while the largest cardinality of a minimal alliance of a graph G is called the *upper alliance number* of G. (Note that the terms alliance number and upper alliance number are used for the cardinalities of minimum defensive alliance and largest minimal defensive alliance of a graph in [HHK00]. In this text, we will use the terms *defensive alliance number* and *upper defensive alliance number* for these parameters). This leads to two invariants for each type of alliance defined in the previous section. Of particular interest are the following invariants:

$a(G) =$ the *defensive alliance number* of graph G

$A(G) =$ the *upper defensive alliance number* of graph G

$\hat{a}(G) =$ the *strong defensive alliance number* of graph G

$\hat{A}(G) =$ the *upper strong defensive alliance number* of graph G

$a^k(G) =$ the *defensive k-alliance number* of graph G

$A^k(G) =$ the *upper defensive k-alliance number* of graph G

$\hat{a}^k(G) =$ the *strong defensive k-alliance number* of graph G

$\hat{A}^k(G) =$ the *upper strong k-defensive alliance number* of graph G

$a_o(G) =$ the *offensive alliance number* of graph G

$A_o(G) =$ the *upper offensive alliance number* of graph G

$\hat{a}_o(G) =$ the *strong offensive alliance number* of graph G

$\hat{A}_o(G) =$ the *upper strong offensive alliance number* of graph G

$\gamma_a(G) =$ the *global defensive alliance number* of graph G

$\gamma_{\hat{a}}(G) =$ the *global strong defensive alliance number* of graph G

$a_p(G) =$ the *powerful alliance number* of graph G

$\hat{a}_p(G) =$ the *strong powerful alliance number* of graph G

$a_s(G) =$ the *secure alliance number* of graph G

From the definitions, it is easy to see that the following relations hold for the above parameters;

i. $a^{-1}(G) = a(G) \leq \hat{a}(G) = a^0(G) \leq \hat{A}(G) = A^0(G)$,

ii. $a(G) \leq A(G) = A^0(G)$,

iii. $a(G) \leq a_d(G)$,

iv. $a(G) \leq \gamma_a(G)$,

v. $\hat{a}(G) \leq \hat{a}_d(G)$,

vi. $\hat{a}(G) \leq \gamma_{\hat{a}}(G)$,

vii. $a_o(G) \leq \hat{a}_o(G) \leq \hat{A}_o(G)$,

viii. $a_o(G) \leq A_o(G)$,

ix. $a_o(G) \leq a_d(G)$,

x. $\hat{a}_o(G) \leq \hat{a}_d(G)$.

2.4 Basic Properties and Known Bounds on Alliance Numbers

The following subsections summarizes several observations and properties of different types of alliances and respective alliance numbers.

2.4.1 Defensive Alliance Numbers

It has been shown in [MGH02] that finding $a(G)$ and $\hat{a}(G)$ for arbitrary graph G is NP-Hard, even when restricted to bipartite or chordal graphs. The classes of graphs for which the values of $a(G)$ and $\hat{a}(G)$ belong to the set $\{1, 2, 3\}$ are summarized below:

Observation 1 *[HHK00]*

i. $a(G) = 1$ *if and only if there exists a vertex* $v \in V$ *such that* $\deg(v) \leq 1$.

ii. $\hat{a}(G) = 1$ *if and only if* G *has an isolated vertex.*

iii. $a(G) = 2$ *if and only if* $\delta(G) \geq 2$ *and* G *has two adjacent vertices of degree at most three.*

iv. $\hat{a}(G) = 2$ *if and only if* $\delta(G) \geq 1$ *and* G *has two adjacent vertices of degree at most two.*

v. $a(G) = 3$ *if and only if* $a(G) \neq 1$, $a(G) \neq 2$, *and* G *has an induced subgraph isomorphic to either (a)* P_3, *with vertices, in order,* u, v, *and* w, *where* $\deg(u)$ *and* $\deg(w)$ *are at most three, and* $\deg(v)$ *is at most five, or (b)* $\mathbf{K}_3$, *each vertex of which has degree at most five.*

vi. $\hat{a}(G) = 3$ *if and only if* $\hat{a}(G) \neq 1$, $\hat{a}(G) \neq 2$, *and* G *has an induced subgraph isomorphic to either (a)* P_3, *with vertices, in order,* u, v, *and* w, *where* $\deg(u)$ *and* $\deg(w)$ *are at most two, and* $\deg(v)$ *is at most four, or (b)* $\mathbf{K}_3$, *each vertex of which has degree at most four.*

The values of defensive alliance numbers for some special classes of graphs are also known and are as follows:

Theorem 2 *[HHK00] For the* $m \times n$ *grid graph* $G_{m,n}$,

i. $a(G_{m,n}) = 1$ *if and only if* $\min\{m, n\} = 1$.

ii. $a(G_{m,n}) = 2$ *if and only if* $\min\{m, n\} \geq 2$.

iii. $\hat{a}(G_{m,n}) = 2$ *if and only if* $\min\{m, n\} < 3$.

iv. $\hat{a}(G_{m,n}) = 3$ *if and only if* $\min\{m, n\} = 3$.

v. $\hat{a}(G_{m,n}) = 4$ *if and only if* $\min\{m, n\} \geq 4$.

Theorem 3 *[HHK00] For any graph* $G = (V, E)$,

i. *if* G *is 1-regular, then* $a(G) = 1$ *and* $\hat{a}(G) = 2$.

ii. *if* G *is 2-regular, then* $a(G) = 2$ *and* $\hat{a}(G) = 2$.

iii. *if* G *is 3-regular, then* $a(G) = 2$ *and* $\hat{a}(G) = girth(G)$.

iv. *if* G *is 4-regular, then* $a(G) = \hat{a}(G) = girth(G)$.

v. *if* G *is 5-regular, then* $a(G) = girth(G)$.

For all the above classes of graphs, the values of defensive alliance numbers are constant, however, for wheels, complete graphs, and complete bipartite graphs, these values can be arbitrarily large. For wheels W_n of order n, $\hat{a}(W_n) = \lceil \frac{n}{2} \rceil$. For the complete graph $\mathbf{K}_n$, $a(\mathbf{K}_n) = \lceil \frac{n}{2} \rceil$ and $\hat{a}(\mathbf{K}_n) = \lfloor \frac{n}{2} \rfloor + 1$. Frick et al [FLH] showed that the complete graphs achieve the upper bound for defensive alliance number $a(G)$.

Theorem 4 *[FLH] For any graph* G *of order* n,

$$a(G) \leq \left\lceil \frac{n}{2} \right\rceil.$$

We now show that the even complete graphs achieve the upper bound for strong defensive alliance number $\hat{a}(G)$, i.e., a minimum strong defensive alliance of graph G has at most $\lfloor \frac{n}{2} \rfloor + 1$ vertices.

Theorem 5 *For any graph G, of order n, $\hat{a}(G) \leq \lfloor \frac{n}{2} \rfloor + 1$.*

Proof. Let A be a minimum defensive 0-alliance of a graph G and $B = V(G) - A$. Assume to the contrary that $|A| > \lfloor \frac{n}{2} \rfloor + 1$. If $\exists T \subseteq B$ and $v \in A$, such that Tor $T \cup \{v\}$ is a defensive 0-alliance then $|T| + 1 \leq \lceil \frac{n}{2} \rceil - 1 < |A|$, a contradiction. Thus, there is a partition $\langle V_1, V_2 \rangle$ of $V(G)$ such that $\forall P \subseteq V_1$, P is not a defensive 0-alliance. Similarly, $\forall Q \subseteq V_2$, Q is not a defensive 0-alliance. Consider such a partition with the property that the size of edge-cutset S separating V_1 and V_2 is minimum among all such partitions. Assume without loss of generality that $|V_1| \geq \lceil \frac{n}{2} \rceil$. Since V_1 is not a defensive 0-alliance, $\exists v \in V_1$ such that $\deg_{V_1}(v) < \deg_{V_2}(v)$. Consider the partition $\langle V_1 - \{v\}, V_2 \cup \{v\} \rangle$. Let S' be the edge-cutset separating $V_1 - \{v\}$ and $V_2 \cup \{v\}$ such that $|S'| = |S| - \deg_{V_2}(v) + \deg_{V_1}(v) < |S|$. Hence, at least one of the sets, $V_1 - \{v\}$ or $V_2 \cup \{v\}$, must be a defensive 0-alliance or contain a subset that is a defensive 0-alliance. Since $V_1 - \{v\}$ is not a defensive 0-alliance, $V_2 \cup \{v\}$ must be a defensive 0-alliance or contain a defensive 0-alliance, but then $|V_2 \cup \{v\}| \leq \lfloor \frac{n}{2} \rfloor + 1 < |A|$, a contradiction. $\square$

2.4.2 Global Defensive Alliance Numbers

We now present some properties of global defensive alliance numbers $\gamma_a(G)$ and $\gamma_{\hat{a}}(G)$. We begin by giving values for specific graph families.

Proposition 6 *[HHH02] For the complete graph K_n,*

(i) $\gamma_a(\mathbf{K}_n) = \lfloor \frac{n+1}{2} \rfloor$, *and*

(ii) $\gamma_{\hat{a}}(\mathbf{K}_n) = \lceil \frac{n+1}{2} \rceil$.

Proposition 7 *[HHH02] For the complete bipartite graph $\mathbf{K}_{r,s}$,*

(i) $\gamma_a(\mathbf{K}_{1,s}) = \lfloor \frac{s}{2} \rfloor + 1$,

(ii) $\gamma_a(\mathbf{K}_{r,s}) = \lfloor \frac{r}{2} \rfloor + \lfloor \frac{s}{2} \rfloor$ *if* $r, s \geq 2$, *and*

(iii) $\gamma_{\hat{a}}(\mathbf{K}_{r,s}) = \lceil \frac{r}{2} \rceil + \lceil \frac{s}{2} \rceil$.

By definition, for every global defensive alliance S, $\partial S = V - S$, i.e., every global defensive alliance set is a dominating set. Hence, $\gamma_a(G) \geq \gamma(G)$, where $\gamma(G)$ is the domination number of graph G.

A set D of vertices of G is defined to be a *total dominating set* if $N(D) = V$. In other words, a total dominating set is a dominating set D with an added condition that every vertex in D must also be adjacent to some other vertex of D. The total domination number $\gamma_t(G)$ of a graph G is the smallest cardinality of a total dominating set. It is easy to see that for any graph G, $\gamma_{\hat{a}}(G) \geq \gamma_t(G)$. In addition, the following lower bounds are known for global defensive alliance numbers.

Theorem 8 *[HHH02] If G is a graph of order n, then*

$$\gamma_a(G) \geq \left(\sqrt{4n+1} - 1\right)/2,$$
$$\gamma_{\hat{a}}(G) \geq \sqrt{n}.$$

Both of the above bounds are tight and are achieved by the graphs $K_k \circ \overline{K}_k$ and $K_k \circ \overline{K}_{k-1}$ respectively, where, for graphs G and H, the *corona* $G \circ H$ is the graph formed from G and $|V(G)|$ copies of H, where the *ith* vertex of G is adjacent to every vertex in the *ith* copy of H.

Theorem 9 *[HHH02] If G is a graph of order n and maximum degree Δ, then*

$$\gamma_a(G) \geq \tfrac{2n}{\Delta+3},$$
$$\gamma_{\hat{a}}(G) \geq \tfrac{2n}{\Delta+2}.$$

Cami et al [CBD04] have shown that the problem of computing $\gamma(G)$ is NP-Hard. A similar construction can be used to show a more general problem of minimum defensive k-alliance is NP-Hard for any fixed k.

2.4.3 Offensive Alliance Numbers

For the offensive alliance numbers, note that every vertex cover is an offensive alliance, and recall that $\alpha_0(G)$ denotes the vertex cover number of G. Thus, we have that $a_0(G) \leq \alpha_0(G)$. In addition, the following bounds on offensive alliance numbers are shown in [FFG02];

Theorem 10 *For all graphs G of order $n \geq 2$, $a_o(G) \leq \frac{2n}{3}$.*

Theorem 11 *For all graphs G of order $n \geq 3$, $\hat{a}_o(G) \leq 5n/6$. Moreover, if G has minimum degree at least 2, then $\hat{a}_o(G) \leq 3n/4$.*

Theorem 12 *For graph G with order n and minimum degree δ, $a_o(G) \leq \hat{a}_o(G) \leq n\,(1/2 + o(\delta))$.*

A tight upper bound or the extremal graphs for the strong offensive alliance numbers are yet unknown.

As is the case with other alliance parameters, computing $a_o(G)$ and $\hat{a}_o(G)$ is also an NP-Hard problem, even for cubic graphs [FFG02]. Similarly, the problem of computing global offensive alliance number is also NP-Hard.

2.4.4 Powerful Alliance Numbers

To illustrate the concept of powerful alliance number $a_p(G)$ and global powerful alliance number $\gamma_{a_p}(G)$, we give values for specific graph families.

Observation 13 *[BDH02]*

i. For the complete graph $\mathbf{K}_n$, $a_p(\mathbf{K}_n) = \gamma_{a_p}(\mathbf{K}_n) = \lceil \frac{n}{2} \rceil$.

ii. For $\mathbf{K}_{r,s}$, $1 \leq r \leq s$, $a_p(\mathbf{K}_{r,s}) = \gamma_{a_p}(\mathbf{K}_{r,s}) = \min\left\{r + \lfloor \frac{s}{2} \rfloor, \lceil \frac{r+1}{2} \rceil + \lceil \frac{s+1}{2} \rceil\right\}$.

iii. For any path P_n, $a_p(P_n) = \gamma_{a_p}(P_n) = \lfloor \frac{2n}{3} \rfloor$.

iv. For any cycle C_n, $a_p(C_n) = \gamma_{a_p}(C_n) = \lceil \frac{2n}{3} \rceil$.

We define a problem PA(POWERFUL ALLIANCE) to be the problem of deciding whether a given graph has a powerful alliance of size less than or equal to a given bound K. Similarly the problem GPA(GLOBAL POWERFUL ALLIANCE) is defined to be the problem of deciding whether a given graph has a global powerful alliance of size less than or equal to a given bound K. It is shown in [CBD04] that GPA is NP-Complete. We now show that the problem PA is also NP-Complete by showing that an NP-Complete variant of GPA is polynomially reducible to PA. The problems we are interested in are formally defined as follows:

GLOBAL POWERFUL ALLIANCE (GPA)

Input: A Graph $G(V, E)$ and a positive integer $K \leq |V|$.

Question: Is there a global powerful alliance in G of size K or less?

AT MOST HALF GLOBAL POWERFUL ALLIANCE (AHGPA)

Input: A Graph $G(V, E)$.

Question: Is there a global powerful alliance in G of size $\frac{|V|}{2}$ or less?

POWERFUL ALLIANCE (PA)

Input: A Graph $G(V, E)$ and a positive integer $K \leq |V|$.

Question: Is there a powerful alliance in G of size K or less?

Theorem 14 *AT MOST HALF GLOBAL POWERFUL ALLIANCE (AHGPA) is NP-Complete.*

Proof. It is easy to see that AHGPA is in NP. Given an instance of GPA, i.e., a graph $G = (V, E)$ and a positive integer $K \leq |V|$, where $V = \{v_1, v_2, \ldots, v_n\}$, we transform the instance of GPA into an instance of AHGPA by constructing a graph $G' = (V', E')$ as follows:

Let $V' = V \cup A_1 \cup A_2 \cup \ldots \cup A_n$, where for $1 \leq i \leq K$, $A_i = \{x_{i,j}, 1 \leq j \leq 11\}$ is a component of 11 vertices, and for $K+1 \leq i \leq n$, $A_i = \{x_{i,j}, 1 \leq j \leq 9\}$ is a component of 9 vertices. Both types of components are shown in Figure 2.1. Thus $|V'| = 10n + 2K$. The vertices $x_{i,1}$ and $x_{i,2}$ of each component A_i are adjacent to the vertex $v_i \in V$. We define E_i to be the set of edges incident to the vertices in A_i, $1 \leq i \leq n$. As shown in the figure, for $i \leq K$,

$E_i = \{x_{i,j}x_{i,k} | 1 \leq j < k \leq 5\} \cup \{x_{i,3}x_{i,6}, x_{i,6}x_{i,7}, x_{i,4}x_{i,8}, x_{i,8}x_{i,9}, x_{i,6}x_{i,10}, x_{i,8}x_{i,11}, v_i x_{i,1}, v_i x_{i,2}\}$,

and for $i > K$, $E_i = \{x_{i,j}x_{i,k} | 1 \leq j < k \leq 5\} \cup \{x_{i,3}x_{i,6}, x_{i,6}x_{i,7}, x_{i,4}x_{i,8}, x_{i,8}x_{i,9}, v_i x_{i,1}, v_i x_{i,2}\}$.

Define the edge set E' of the constructed graph G' as

$$E' = E \cup \left(\bigcup_{1 \leq i \leq n} E_i \right).$$

We now claim that the constructed graph G' has a global powerful alliance of size less than or equal to $\frac{|V'|}{2}$ if and only if the given graph G has a global powerful alliance of size less than or equal to K. The proof of the claim is as follows:

$\Longrightarrow$ Suppose that the given graph G has a global powerful alliance S of size less than or equal to K. Consider a set $T = S \cup \left(\bigcup_{1 \leq i \leq n} \{x_{i,2}, x_{i,3}, x_{i,4}, x_{i,6}, x_{i,8}\} \right)$. Since S is a global powerful alliance in G, for each $v_i \in V$, $|N_S[v_i]| \geq |N_{V-S}[v_i]|$. By construction, for each $v_i \in V$, $N_{T-V}[v_i] = \{x_{i,2}\}$ and $N_{V'-V-T}[v_i] = \{x_{i,1}\}$. Hence, $|N_T[v_i]| = |N_S[v_i]| + 1 \geq$

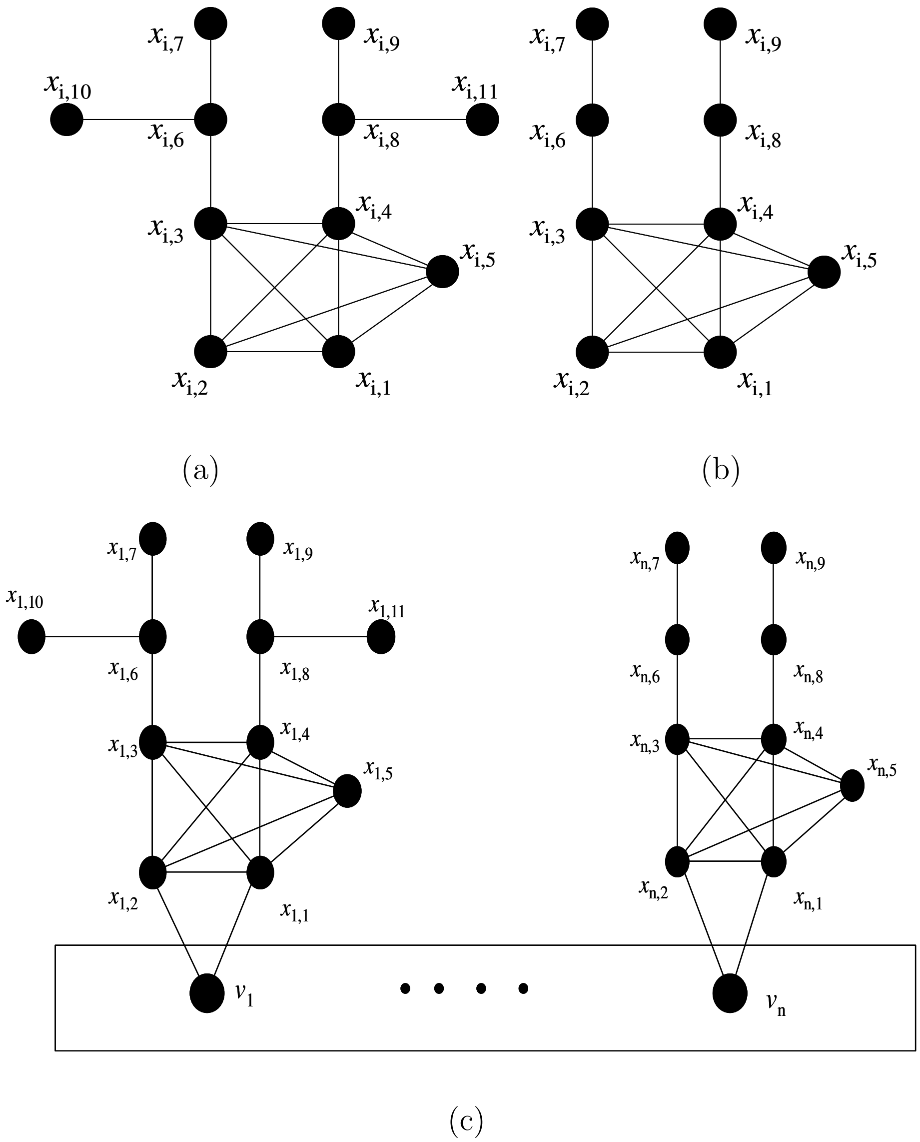


Figure 2.1: (a) An 11-vertex component (b)A 9-vertex component.(c) Constructed graph G'. Each vertex v_i, $1 \leq i \leq K$ is connected to an 11-vertex component and each vertex v_j, $K+1 \leq j \leq n$, is connected to a 9-vertex component.

$|N_{V-S}[v_i]| + 1 = |N_{V'-T}[v_i]|$. Furthermore, for all $x \in \bigcup_{1 \le i \le n} A_i$, $|N_T[x]| \ge |N_{V'-T}[x]|$. Thus, T is a powerful alliance and $|T| \le 5n + K = \frac{|V'|}{2}$.

$\Longleftarrow$ Let S' be a global powerful alliance of the constructed graph G', such that $|S'| \le \frac{|V'|}{2} = 5n + K$. From the construction of graph G', it is easy to see that any global powerful alliance must contain at least five vertices from each A_i, $1 \le i \le n$. Thus $|S' \cap V| \le K$. Let $S = S' \cap V$ and $W_{S'} = \{v_i | N_S[v_i] < N_{V-S}[v_i]\}$. Let S' be a minimum global powerful alliance in graph G', such that $|W_{S'}|$ is minimum among all such alliances.

Suppose now that $W_{S'} \neq \emptyset$ and let $v_i \in W_{S'}$. Since S' is a global powerful alliance, we must have $\{x_{i,1}, x_{i,2}\} \subset S'$ and $2 \le |N_{S'}[v_i]| = |N_{V'-S'}[v_i]| = |N_{V-S'}[v_i]|$. Also, by the design of component A_i and the definition of global powerful alliance, $|S' \cap A_i| \ge 6$. Arbitrarily pick a vertex $v_j \in N_{V-S'}(v_i)$ and consider the set $T' = (S' - A_i) \cup \{x_{i,2}, x_{i,3}, x_{i,4}, x_{i,6}, x_{i,8}\} \cup \{v_j\}$. Note that all the vertices in $V' - A_i$ have equal or more neighbors (including themselves) in T' than they had in S'. Also, for all vertices $a \in A_i$, $N_{T'}[a] \ge N_{V'-T'}[a]$. Hence, T' is a minimum global powerful alliance in graph G' and $W_{T'} = W_{S'} - \{v_i\}$, which is contrary to $W_{S'}$ being a minimum such set. Hence $W_{S'} = \emptyset$, i.e., for all i, $N_S[v_i] \ge N_{V-S}[v_i]$, which implies that S is a global powerful alliance in graph G. $\square$

Now that we have shown that AHGPA is NP-Complete, we prove that POWERFUL ALLIANCE (PA) is also NP-Complete.

Theorem 15 *POWERFUL ALLIANCE (PA) is NP-Complete.*

Proof. It is easy to see that PA is in NP. Given an instance of AHGPA, i.e., a graph $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$, we transform the instance of AHGPA into an instance of PA by setting $K' = \lfloor \frac{3n}{2} \rfloor + 2$ and constructing a graph $G' = (V', E')$ as follows:

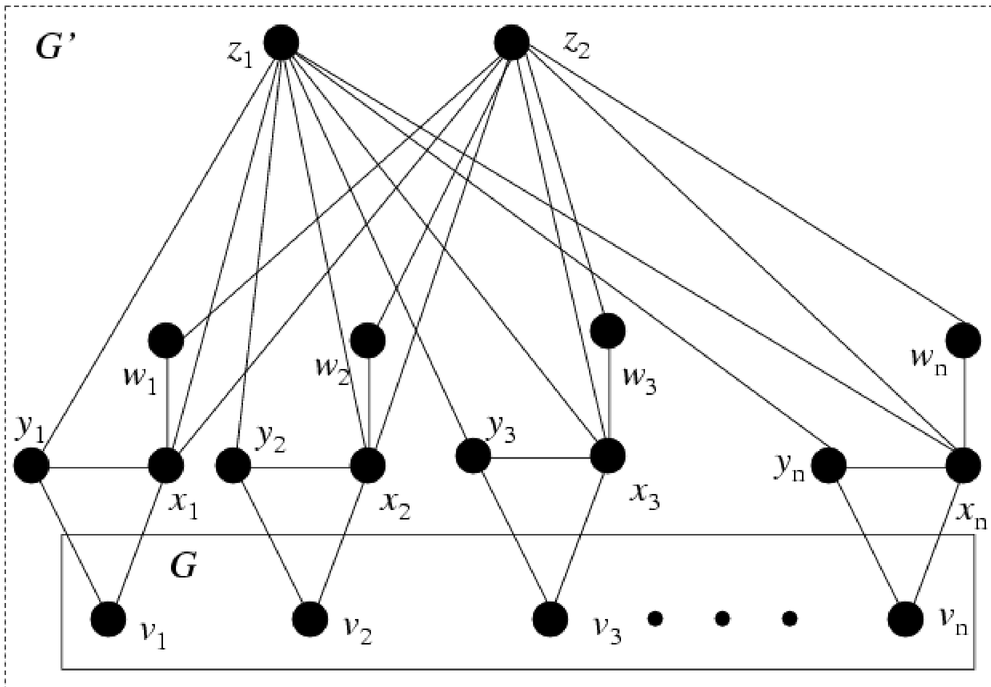


Figure 2.2: (a) Construction of an instance of PA from an instance of AHGPA.

The vertex set V' of the graph G' is defined as $V' = V \cup W \cup X \cup Y \cup \{z_1, z_2\}$, where $W = \{w_1, w_2, \ldots, w_n\}$, $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$. W, X and Y are independent sets, such that for all $w_i \in W$, $N(w_i) = \{x_i, z_2\}$, for all $x_i \in X$, $N(x_i) = \{v_i, w_i, y_i, z_1, z_2\}$, and for all $y_i \in Y$, $N(y_i) = \{v_i, x_i, z_1\}$. Also, $N(z_1) = X \cup Y$ and $N(z_2) = X \cup W$. (See Figure 2.2). Formally, the edge set E' of the constructed graph G', is defined as

$$E' = E \cup \left(\bigcup_{1 \le i \le n} \{w_i x_i, w_i z_2, x_i v_i, x_i y_i, x_i z_1, x_i z_2, y_i v_i, y_i z_1\} \right)$$

The order of the constructed graph, $|V'| = 4n+2$ and the size of the graph, $|E'| = |E|+8n$, which are polynomially related to the size of the AHGPA problem. We now claim that the constructed graph G' has a powerful alliance of size less than or equal to $K' = \lfloor \frac{3n}{2} \rfloor + 2$ if and only if the given graph G has a global powerful alliance of size less than or equal to $\frac{n}{2}$.

$\Longrightarrow$ Suppose that the given graph G has a global powerful alliance S of size less than or equal to $\frac{n}{2}$. Let $S = \{v_1, v_2, \ldots, v_r\}$, $r \leq \frac{n}{2}$. Consider a set $T = S \cup X \cup \{z_1, z_2\}$. Since S is a global powerful alliance in G, for each $v_i \in V$, $|N_S[v_i]| \geq |N_{V-S}[v_i]|$. By construction, for each $v_i \in V$, $|N_{T-V}[v_i]| = 1$ and $|N_{V'-V-T}[v_i]| = 1$. Hence, $|N_T[v_i]| = |N_S[v_i]| + 1 \geq |N_{V-S}[v_i]| + 1 = |N_{V'-T}[v_i]|$. Similarly, for all vertices $x_i \in X$, $|N_T[x_i]| \geq 3 \geq |N_{V'-T}[x_i]|$. For all $y_i \in Y$, $|N_T[y_i]| \geq 2 \geq |N_{V'-T}[y_i]|$. For all $w_i \in W$, $|N_T[w_i]| = 2 > |N_{V'-T}[w_i]| = 1$. Finally, $|N_T[z_1]| = n + 1 > |N_{V'-T}[z_1]| = n$ and $|N_T[z_2]| = n + 1 > |N_{V'-T}[z_2]| = n$. Since for all vertices $v \in N[T]$, $|N_T[v]| \geq |N_{V'-T}[v]$, T is a powerful alliance in graph G' and $|T| = r + n + 2 \leq \lfloor \frac{3n}{2} \rfloor + 2 = K'$.

$\Longleftarrow$ Suppose that the constructed graph G' has a powerful alliance of size less than or equal to $K' = \lfloor \frac{3n}{2} \rfloor + 2$. We now present a sequence of results, which culminate with the proof that the graph G has a global powerful alliance of size less than or equal to $\frac{n}{2}$.

Lemma 16 *If S' is a powerful alliance in graph G' such that $|S'| \leq \lfloor \frac{3n}{2} \rfloor + 2$, then $\{z_1, z_2\} \subseteq S'$.*

Proof. Assume to the contrary, and first let $S' - V = \emptyset$ and let $S' = \{v_1, v_2, \ldots, v_r\}$. Then for all x_i, $1 \leq i \leq r$, $|N_{S'}[x_i]| = 1 < |N_{V'-S'}[x_i]| = 5$, which is contrary to S' being a powerful set. Thus, $S' - V \neq \emptyset$. Since for all $u \in V' - V$, $\{z_1, z_2\} \cap N[u] \neq \emptyset$, $\{z_1, z_2\} \cap N[S'] \neq \emptyset$.

We now consider two exhaustive cases:

Case 1: $S' \cap \{z_1, z_2\} = \emptyset$. Consider $z_i \in \{z_1, z_2\} \cap N[S']$. By the definition of powerful alliance, $|N_{S'}(z_i)| \geq |N_{V'-S'}[z_i]|$. From the construction, $|N[z_i]| = 2n+1$, therefore we have, $|N_{S'}(z_i)| \geq n+1$. Let $X' = \{x_i | \{x_i, y_i\} \cap N_{S'}(z_i) \neq \emptyset\}$. Since $N(z_i) = \bigcup_{1 \leq i \leq n} \{x_i, y_i\}$, $|X'| \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$. Also note that for all $x_i \in X'$, $|N[x_i]| = 6$, hence, we must have $|N_{S'}[x_i]| = |\{v_i, w_i, x_i, y_i, z_1, z_2\} \cap S'| \geq 3$, which implies that, for all $x_i \in X'$, $|\{v_i, w_i, x_i, y_i\} \cap S'| \geq 3$. Thus $|S'| \geq 3|X'| = 3\left\lfloor \frac{n}{2} \right\rfloor + 3 > K'$, a contradiction.

Case 2: $|S' \cap \{z_1, z_2\}| = 1$. Since for all $x_i \in X$, $\{z_1, z_2\} \subset N[x_i]$, $X \subset N[S']$. Hence, we must have $|N_{S'}[x_i]| \geq 3$. That is, for all $x_i \in X$, $|\{v_i, w_i, x_i, y_i\} \cap S'| \geq 2$, which implies that $|S'| \geq 2|X| + 1 = 2n + 1 > K'$, again a contradiction.

Since both cases lead to contradiction, we must assume that $\{z_1, z_2\} \subseteq S'$. $\square$

Corollary 17 *If S' is a powerful alliance in graph G' such that $|S'| \leq \left\lfloor \frac{3n}{2} \right\rfloor + 2$, then $|S' - V| \geq n + 2$.*

Corollary 18 *If S' is a powerful alliance in graph G' such that $|S'| \leq \left\lfloor \frac{3n}{2} \right\rfloor + 2$, then $(V' - V) \subseteq N[S']$.*

Lemma 19 *If S' is a powerful alliance in graph G' such that $|S'| \leq \left\lfloor \frac{3n}{2} \right\rfloor + 2$, then for all i, $1 \leq i \leq n$, $S' \cap \{w_i, x_i\} \neq \emptyset$.*

Proof. From Corollary 18, $W \subset N[S']$. Since for all $w_i \in W$, $N[w_i] = \{w_i, x_i, z_2\}$, by the definition of power alliance, $|N_{S'}[w_i]| \geq 2$, i.e., $|S' \cap \{w_i, x_i\}| \geq 1$. $\square$

Lemma 20 *If S' is a powerful alliance in graph G' such that $|S'| \leq \lfloor \frac{3n}{2} \rfloor + 2$, then for all i, $1 \leq i \leq n$, $S' \cap \{v_i, x_i, y_i\} \neq \emptyset$.*

Proof. From Corollary 18, $Y \subset N[S']$. Since for all $y_i \in Y$, $N[y_i] = \{v_i, x_i, y_i, z_1\}$, by the definition of power alliance, $|N_{S'}[y_i]| \geq 2$, i.e., $|S' \cap \{v_i, x_i, y_i\}| \geq 1$. □

Corollary 21 *If S' is a powerful alliance in graph G' such that $|S'| \leq \lfloor \frac{3n}{2} \rfloor + 2$, then $V \subset N[S']$.*

Corollary 22 *If S' is a powerful alliance in graph G' such that $|S'| \leq \lfloor \frac{3n}{2} \rfloor + 2$, then S' is a global powerful alliance.*

Lemma 23 *If S' is a powerful alliance in graph G' such that $|S'| \leq \lfloor \frac{3n}{2} \rfloor + 2$, then $S' \cap V$ is a global powerful alliance in graph G.*

Proof. Let S' be a powerful alliance of the constructed graph G', such that $|S'| \leq \lfloor \frac{3n}{2} \rfloor + 2$. Let $S = S' \cap V$ and $U_{S'} = \{v_i | N_S[v_i] < N_{V-S}[v_i]\}$. Let S' be a powerful alliance for which $|U_{S'}|$ is minimum among all such powerful alliances in the graph G' of size less than or equal to $\lfloor \frac{3n}{2} \rfloor + 2$. If $U_{S'} = \emptyset$ then $S' \cap V$ is a global powerful alliance in graph G.

Suppose now that $U_{S'} \neq \emptyset$. Let $v_i \in U_{S'}$. From Corollary 22, S' is a global powerful alliance, hence, we must have $\{x_i, y_i\} \subset S'$ and $2 \leq |N_{S'}[v_i]| = |N_{V'-S'}[v_i]| = |N_{V-S'}[v_i]|$. Arbitrarily pick a vertex $v_j \in N_{V-S'}(v_i)$ and consider the set $T' = (S' - \{y_i\}) \cup \{v_j\}$. Note that for all $u \in V' - \{x_i, y_i, z_1\}$, $|N_{T'}[u]| \geq |N_{S'}[u]|$. Also, $|N_{T'}[x_i]| \geq 4 > |N_{V'-T'}[x_i]|$ and $|N_{T'}[y_i]| = 3 > |N_{V'-T'}[y_i]|$. Now there are two cases:

Case 1: $|N_{T'}[z_1]| \geq |N_{V'-T'}[z_1]|$. Then for all vertices $u \in V'$, $N_{T'}[u] \geq N_{V'-T'}[u]$, i.e., T' is a powerful alliance. In addition, $|T'| = |S'|$, and $|U_{T'}| < |U_{S'}|$, a contradiction.

Case 2: $|N_{T'}[z_1]| < |N_{V'-T'}[z_1]|$. From Lemma 17, we have, $|N_{T'}[z_1]| = n$. Since $z_1 \in T'$, by pigeonhole principle, there exists a set $\{x_k, y_k\}$ such that $\{x_k, y_k\} \cap T' = \emptyset$. From Lemma 19, $w_k \in T'$. Let $T = (T' - \{w_k\}) \cup \{x_k\}$. It is easy to see that for all the vertices $u \in V'$, $N_T[u] \geq N_{V'-T}[u]$. Hence T is a powerful alliance in graph G' with $|T| = |T'| = |S'|$, and $|U_T| < |U_{S'}|$, a contradiction.

Since both cases lead to contradiction, we must conclude that our initial assumption that $U_{S'} \neq \emptyset$ was incorrect. Thus, $S' \cap V$ is a global powerful alliance in graph G. $\square$

It follows from Corollaries 17 and 22, and from Lemma 23 that if the constructed graph G' has a powerful alliance of size less than or equal to $K' = \left\lfloor \frac{3n}{2} \right\rfloor + 2$, then the graph G has a global powerful alliance of size less than or equal to $\frac{n}{2}$. $\square$

2.5 Open Problems

We conclude this chapter with a list of open problems relating to alliances and alliance numbers.

- Determine the relationships between alliance numbers (defensive, offensive, global, etc.) and other domination parameters.

- Find the real upper bound for the offensive alliance numbers and the extremal graphs.

- Characterize the graphs (or some family of graphs) for which $\gamma(G) = \gamma_a(G)$.

- Characterize the graphs (or some family of graphs) for which $\gamma_t(G) = \gamma_{\hat{a}}(G)$.

- Characterize the graphs (or some family of graphs) for which $a_o(G) = \hat{a}_o(G)$.

- Characterize the graphs (or some family of graphs) for which $a_o(G) = \alpha_o(G)$.

- Determine the computational complexity of computing the parameters $A(G)$, $A_o(G)$, $a_d(G)$, $\hat{a}_d(G)$, $\gamma_a(G)$, and $\gamma_{\hat{a}}(G)$.

- Study the alliance numbers for $k-$alliances and p-alliances.

- Study the global counterparts for alliances other than defensive alliances.

- Determine the exact values or good bounds for special families of graphs (e.g., trees, grid graphs, planar, outer-planar graphs).

- Given a graph G and a vertex $v \in V$, define the alliance number (of some type) of v, $a(v)$ to be the smallest alliance (of that type) containing vertex v. What is the complexity of finding $a(v)$ (for each each type of alliance)?

- Given a graph G and a set $S \in V$, what is $a(S)$, that is the smallest cardinality of an alliance containing set S (for each type of alliance)?

- Given a graph G, define *alliance packing numbers* $P_a(G)$ to equal the maximum number of pairwise-disjoint, alliances contained in G. Similarly, define *alliance partitioning* *numbers*, $\psi_a(G)$, to equal the maximum order of a partition $\Pi = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$, such that each block of the partition V_i is an alliance. What is the complexity of finding $P_a(G)$ and $\psi_a(G)$ for each type of alliance.

- Find exact efficient algorithms for computing the alliance numbers that are not NP-Hard.
- Find the approximate algorithms for the alliance numbers that are NP-Hard.
- What is the minimum error that can be guaranteed to compute the alliance numbers in polynomial time?

CHAPTER 3

PARTITIONING A GRAPH INTO DEFENSIVE AND GLOBAL DEFENSIVE ALLIANCES

3.1 Introduction

In this chapter, we discuss the problem of partitioning a graph into defensive and strong defensive alliances. The problem of partitioning a graph into strong defensive alliances was first introduced by Gerber and Kobler [GK00] and was referred to as "**Satisfactory Graph Partitioning Problem (SGP)**".

Consider a graph $G = (V, E)$ without loops or multiple edges. Recall from chapter 2 that a vertex v in set $A \subseteq V$ is said to be k-satisfied with respect to A if $\deg_A(v) \geq \deg_{V-A}(v) + k$, where $\deg_A(v) = |N(v) \cap A| = |N_A(v)| = \deg(v) - \deg_{V-A}(v)$. Also recall that a set A is a defensive k- *alliance* if all vertices in A are k-satisfied with respect to A. Note that a defensive (-1)-alliance is a "defensive alliance" (as defined in [HHK00]), and a defensive 0-alliance is a "strong defensive alliance" or "cohesive set" [SD02a]. A k-defensive alliance

A is called *global* if every vertex in $V - A$ is adjacent to at least one member of the alliance A.

A graph is said to be k-***satisfiable*** if there exists a vertex partition into two or more nonempty sets so that every vertex is k-satisfied with respect to the set in which it occurs, i.e., a partition into two or more k-defensive alliances (it is called $k-$***unsatisfiable*** otherwise). Such a partition is referred to as k-***satisfactory partition***.

Our problem, the k- *Satisfactory Graph Partitioning problem* (k-SGP), consists in determining if a graph is k-satisfiable or not, i.e., whether a given graph can be partitioned into two k-defensive alliances. The problem can be easily generalized to other types of alliances. Of particular interest are weighted defensive $k-$alliances and weighted defensive p-alliances.

A related problem has been considered in Artificial Intelligence to study a neural network model of the human brain known as *binary coherent system* (BCS)[Hop82] or *stable configuration problem*[SY91]. The problem can be formally stated as follows: Given an edge weighted directed graph $G = (V, E)$ and a threshold value t_v for each vertex $v \in V$. Find a partition $\langle V_{-1}, V_{+1} \rangle$ of V such that for every vertex v, the energy $E(v)$ is non-negative, where,

$$E(v) = s_v \left(t_v + \sum_{e=(u,v) \in E} w_e s_u \right)$$

$s_v = 1$, if $v \in V_{+1}$ and $s_v = -1$, otherwise.

Note that BCS problem allows a set in a partition to be empty, while SGP does not. The BCS problem has a polynomial time sequential algorithm if all of the weights and thresholds are input in unary [Lub86].

The *Different Than Majority Labelling* (DTML) problem [Lub86] is a special case of the BCS problem. Here the threshold value t_v is 0 for every vertex v, and all edge weights are -1. The DTML problem may also be viewed as a similar but complementary graph partitioning problem of 0-SGP known as *Unfriendly Graph Partitioning Problem (UGP)* [BK], where a partition is said to be unfriendly if each vertex has as many or more neighbors outside the set in which it occurs than inside it. While there exists an unfriendly graph partition for every graph[1], this is not the case for satisfactory partitions of vertices. For example, complete graphs $\mathbf{K}_n$ and complete bipartite graphs $\mathbf{K}_{p,q}$ (when p or q is odd) are not 0-satisfiable. Similarly, odd complete graphs are not (-1)-satisfiable. There exists a polynomial time algorithm for finding an unfriendly partition for graphs. On the other hand, the problem k-SGP, $k \geq 0$, was also shown to be NP-Hard for unweighted graphs in [BTV03a, BTV03b]. For $k < -1$, every graph has a k-satisfactory partition[Sti96], and such a partition can be found in polynomial time[BTV03b].

Another complementary problem of *SGP* is that of partitioning the vertex set into two or more sets such that none of these sets contain any k-alliance, i.e., a partition into k-alliance-free sets. The existence of such a partition is again not guaranteed, for example

[1] All finite graphs have unfriendly bipartitions, but there exist infinite graph with no unfriendly bipartition [SM90]. However, all graphs have an unfriendly 3-partition

complete graphs of odd order and odd cycles do not have a partition into $0-$alliance free sets. However, we have characterized the graphs that have such a partition[SD04].

In this chapter, we present results on the solution and the complexity of the following problems.

PARTITION INTO DEFENSIVE ALLIANCES ((−1)-SGP)

Input: A Graph $G(V, E)$.

Question: Is there a partition $\langle V_1, V_2\rangle$ of V, such that both V_1 and V_2 are defensive alliances ((−1)-defensive alliances)

PARTITION INTO STRONG DEFENSIVE ALLIANCES (0-SGP)

Input: A Graph $G(V, E)$.

Question: Is there a partition $\langle V_1, V_2\rangle$ of V, such that both V_1 and V_2 are strong defensive alliances (0-defensive alliances)

PARTITION INTO GLOBAL DEFENSIVE ALLIANCES

Input: A Graph $G(V, E)$.

Question: Is there a partition $\langle V_1, V_2\rangle$ of V, such that both V_1 and V_2 are global defensive alliances (global (−1)-defensive alliances)

PARTITION INTO GLOBAL STRONG DEFENSIVE ALLIANCES

Input: A Graph $G(V, E)$.

Question: Is there a partition $\langle V_1, V_2\rangle$ of V, such that both V_1 and V_2 are global strong defensive alliances (global 0-defensive alliances)

The organization of this chapter is as follows. In Section 2, we present some basic observations. Section 3 discusses the relationship between satisfiability and connectivity of graphs. Section 4 presents results regarding categorization of satisfiable graph by their subgraphs. Section 5 treats special cases, for example, Eulerian graphs, regular graphs and line graphs. Section 6 concludes the chapter.

Since, disconnected graphs are trivially satisfiable, we will only consider connected graphs.

3.2 Basic Properties

Since every defensive k-alliance is also a defensive l-alliance, for all $l < k$ and since every global defensive k-alliance is also a defensive k-alliance, our first observation is immediate.

Observation 24 *For any graph G*

(i) *If G has a k-satisfactory partition then G has an l-satisfactory partition, for all $l < k$.*

(ii) *If G has a partition into global defensive k-alliances then G has a k-satisfactory partition.*

Also, since for an Eulerian graph a $(2r-1)$-defensive alliance is also a $2r$-defensive alliance, we have,

Observation 25 *For an Eulerian graph G and $r \leq \frac{\delta(G)}{2}$, a partition into (global) $(2r-1)$-defensive alliances is also a partition into (global) $2r$-defensive alliances.*

Since $V(G)$ is itself a defensive k-alliance, $k \leq \delta(G)$, we define a defensive k-alliance $X \subset V$ to be *locally maximal* if $\forall v \notin X$, $X \cup \{v\}$ is not a defensive k-alliance. If X is a locally maximal defensive k-alliance of graph G then $V(G) - X$ is a defensive $(1-k)$-alliance.

Proposition 26 *For $k \leq 0$, a graph G is k-satisfiable if it has a locally maximal defensive k-alliance.*

The converse of the above proposition is not true, for example, $\mathbf{C}_n, \forall n > 3$ is 0-satisfiable but has no locally maximal defensive 0-alliance.

Similarly, a *locally minimal* defensive k-alliance is a defensive k-alliance X, such that $\forall v \in X$, $X - \{v\}$ is not a defensive k-alliance. Every minimal defensive k-alliance is also a locally minimal defensive k-alliance but a locally minimal defensive k-alliance need not be a minimal defensive k-alliance. A *minimum defensive k-alliance* is a minimal defensive k-alliance of smallest order. If a graph G is k-satisfiable, then, by definition, it has at least two disjoint minimal defensive k-alliances (the converse of this is also true and is Lemma 27).

Lemma 27 *[Sti96] For $k \leq 0$, a graph G is k-satisfiable if and only if it has two disjoint k-alliances.*

Hence, if every minimal defensive k-alliance of a graph G has at least $\lfloor \frac{n}{2} \rfloor + 1$ vertices then G is k-unsatisfiable. From Theorems 4 and 5, we know that a minimum defensive (-1)-alliance of a graph has at most $\lceil \frac{n}{2} \rceil$ vertices, whereas a minimum defensive (0)-alliance has no more than $\lfloor \frac{n}{2} \rfloor + 1$ vertices.

Next we present the satisfiability of some common graph families.

Observation 28 *The following graphs have partition into strong defensive alliances (i.e., they are* 0*-satisfiable):*

(i) *Complete graphs of even order minus a 1-factor.*

(ii) *Complete bipartite graphs* $K_{p,q}$ *if both* p *and* q *are even.*

(iii) *Grid graphs.*

(iv) *Cycles of order greater than 3.*

(v) *Separable graphs and graphs that have a bridge, which is not a pendant edge, for example, trees with diameter greater than 2.*

The first two of the above graphs also have a partition into global strong defensive alliances. From Observation 24, all the above graphs also have a partition into defensive alliances. Examples of graphs that have a partition into defensive alliances are presented in the next observation.

Observation 29 *The following graphs have partition into defensive alliances (i.e., they are* (-1)*-satisfiable):*

(i) *Complete graphs of even order.*

(ii) *Complete bipartite graphs.*

(iii) *Graphs that have one or more pendant vertices.*

The following result was shown in [GK01]:

Theorem 30 *[GK01] Every graph (that is not $K_{1,n}$) of girth at least 5 is* 0*-satisfiable.*

3.3 Satisfiability and Connectivity

In this section, we discuss the relation between the connectivity and satisfiability of a graph. We know that complete graphs are 0-unsatisfiable and that trees, except stars, are 0-satisfiable. We first ask if there is a bound for which graphs with minimum degree greater than this bound are k-unsatisfiable, for $k \in \{-1, 0\}$. We prove next that no such bound exists.

Theorem 31 *There is no $r \in [0,1)$ such that $\delta(G) \geq rn \Rightarrow G$ is* 0*-unsatisfiable.*

Proof. Note that $\forall p \geq 1, \mathbf{K}_{2p}$ minus 1-factor is 0-satisfiable, where V_1 and V_2 form a 0-satisfactory partition such that $G[V_1] \cong G[V_2] \cong \mathbf{K}_p$. Assume to the contrary that such an r exists. Consider $p \geq \frac{1}{1-r}$, and let $G \cong \mathbf{K}_{2p}$ minus a 1-factor. Then $\delta(G) = 2p - 2$. Since G is 0-satisfiable, therefore by assumption, $2p - 2 < r(2p) \Rightarrow p < \frac{1}{1-r}$, hence a contradiction. □

Similarly it can be proved that there is no $r \in [0,1)$ such that $density(G) \geq r \Rightarrow G$ is 0-unsatisfiable, where $density(G) = \frac{|E|}{n(n-1)/2}$.

We define a *Critical Cutset* $S = \langle V_1, V_2 \rangle$ of a connected graph G to be a minimal cutset, such that $|V_i| > 1,\ i \in \{1,\ 2\}$ and moving any vertex from one set to the other does not decrease the size of the resulting cutset.

Theorem 32 *G is 0-satisfiable if and only if it has a critical cutset.*

Proof. Suppose G has a critical cutset $S = \langle V_1, V_2 \rangle$ and there exists a vertex v which is not 0-satisfied. Assume without loss of generality that $v \in V_1$. Then $\deg_{V_1}(v) < \deg_{V_2}(v)$ and we may form a new partition $S' = \langle V_1 - \{v\}, V_2 \cup \{v\} \rangle$ where $|V_1 - \{v\}| \geq 1$. Now, $|S'| = |S| - \deg_{V_2}(v) + \deg_{V_1}(v)$. Since $\deg_{V_1}(v) < \deg_{V_2}(v)$, we must have that $|S'| < |S|$, contradicting the assumption that S is a critical cutset of G.

For the converse, consider a 0-satisfiable graph G such that the cutset $S = \langle V_1, V_2 \rangle$ forms a 0-satisfactory partition. Suppose that S is not a critical cutset, that is, there exists a vertex v, such that moving v from one set of the partition to another would decrease the size of cutset. Assume without loss of generality that $v \in V_1$. Then $S' = \langle V_1 - \{v\}, V_2 \cup \{v\} \rangle$ and $|S'| < |S|$. But $|S'| = |S| - \deg_{V_2}(v) + \deg_{V_1}(v)$ which means that $\deg_{V_1}(v) < \deg_{V_2}(v)$ and contradicts the assumption that $S = \langle V_1, V_2 \rangle$ is a 0-satisfactory partition. $\square$

Recall that *edge connectivity* $\kappa_1(G)$ of a graph G is the minimum number of edges whose removal from G results in a disconnected graph. The following result, also proven in [GK00], is a direct consequence of Theorem 32.

Corollary 33 *A connected graph G is 0-satisfiable if $\kappa_1(G) < \delta(G)$.*

The next theorem provides the relation between 0-satisfiability and vertex connectivity $\kappa(G)$;

Theorem 34 *For $k \leq 0$, a graph G is k-satisfiable if $\kappa(G) \leq \left\lfloor \frac{\delta(G)-k}{2} \right\rfloor$.*

Proof. Suppose for a graph G, that $\kappa(G) \leq \left\lfloor \frac{\delta(G)-k}{2} \right\rfloor$ and G is k-unsatisfiable. From Corollary 33, we may assume G is connected and that V' is a set of disconnecting vertices of G such that $1 \leq |V'| \leq \left\lfloor \frac{\delta(G)-k}{2} \right\rfloor$. Let A be the set of vertices of one of the components of $G - V'$ and let $B = V - V' - A$. The edge cutset $S = \langle B, A \cup V' \rangle$ partitions V into two subsets. Since $\forall v \in B,\ N(v) \cap A = \emptyset$, $N(v) - B \subseteq V'$, and thus $\deg_{V-B}(v) \leq \left\lfloor \frac{\delta(G)-k}{2} \right\rfloor \leq \deg_B(v) - k$. Hence, every vertex of B is k-satisfied. The only vertices in G, which may not be k-satisfied with respect to the partition $\langle B, A \cup V' \rangle$, are those in V'. Now perform the following procedure on the partition.

While $\exists v \in V'$ such that $\deg_{V-B}(v) < \deg_B(v) + k$

Begin

Set $B \leftarrow B \cup \{v\}, V' \leftarrow V' - \{v\}$

End

This procedure will certainly terminate, as there is only a finite number of elements in V' and vertices are only moved from set V' to set B. Since every vertex of B was initially k-satisfied, no vertices are removed from B. Therefore every vertex of B is still k-satisfied. Also, all vertices of V' are now k-satisfied. Since at most $\left\lfloor \frac{\delta(G)-k}{2} \right\rfloor$ vertices were moved from set V' to set B, vertices of A are each adjacent to at most $\left\lfloor \frac{\delta(G)-k}{2} \right\rfloor$ vertices in B, and are k-satisfied. Thus, G is k-satisfiable. □

3.4 Subgraph Characterizations

In this section we show that there is no forbidden subgraph characterization of minimal defensive k-alliances. We also show that the same holds for satisfiable and unsatisfiable graphs.

To show the nonexistence of a forbidden subgraph characterization for satisfiable graphs, we first prove that for $k \leq 0$, there is no such characterization for minimal defensive k-alliances.

Lemma 35 *For $k \leq \delta(G)$, there is no forbidden subgraph characterization for subgraphs induced by minimal defensive k-alliances.*

Proof. Suppose to the contrary that $G = (V, E)$ is a forbidden subgraph for graphs induced by minimal defensive k-alliances. Since minimal alliances are connected, we may assume that G is connected. Let $V = \{v_1, v_2, \ldots, v_n\}$, and construct a graph $G' = (V', E')$ as follows:

$V' = V \cup X_1 \cup X_2 \cup \ldots \cup X_n$, where $X_i = \left\{x_1^{(i)}, x_2^{(i)}, \ldots, x_{\deg(v_i)-k}^{(i)}\right\}$ is a set of independent vertices; and $E' = E \cup Y_1 \cup Y_2 \cup \ldots \cup Y_n$, where $Y_i = \left\{v_i x_1^{(i)}, v_i x_2^{(i)}, \ldots, v_i x_{\deg(v_i)-k}^{(i)}\right\}$. Hence, by construction, $\forall v \in V, \deg_V(v) = \deg_{V'-V}(v) + k$. Since G is connected, V is a minimal k-alliance of graph G', contradicting our initial assumption. □

Theorem 36 *For $k \leq \delta(G)$, there is no forbidden subgraph characterization of k-satisfiable graphs.*

Proof. Suppose to the contrary that $G = (V, E)$ is a forbidden subgraph for graphs induced by k-satisfiable graphs. Hence, G cannot be induced by any subset of a k-satisfiable graph. Construct a graph $G' = (V', E')$ as in the proof of Lemma 35, such that V is a minimal k-alliance of G'. Add edges between all the vertices in $V' - V$. From construction, $|V' - V| = 2|E| - kn$. Hence $V' - V$ forms a clique of $2|E| - kn$ vertices and hence is a defensive k-alliance. Thus, V and $V' - V$ is a k-satisfactory partition of graph G', a contradiction. □

Note that the graph constructed in the above proof has a partition into global defensive k-alliances. Thus we have:

Corollary 37 *For $k \leq \delta(G)$, there is no forbidden subgraph characterization of the graphs having a partition into global defensive k-alliances.*

The join of simple graphs G_1 and G_2, written $G_1 \vee G_2$, is the graph obtained by adding the edges $\{xy : x \in V(G_1), y \in V(G_2)\}$.

Theorem 38 *For $k \geq 0$, there is no forbidden subgraph characterization of k-unsatisfiable graphs.*

Proof. Suppose to the contrary that $G = (V, E)$ is a forbidden subgraph for k-unsatisfiable graphs, that is G cannot be an induced subgraph of any k-unsatisfiable graph.

We construct a graph $G' = G \vee K_{n-k+1}$ where n is the number of vertices in G. Therefore, G' must be k-satisfiable, since G is an induced subgraph of G'. Let $\langle A, B \rangle$ be a k-satisfactory

partition of G' and consider $v \in V(K_{n-k+1})$. Assume, without loss of generality, that $v \in A$. Since $\deg(v) = 2n-k$, $|A| \geq n+1$. Then $|B| \leq n-k$ and no vertex of $V(K_{n-k+1})$ can be satisfied in B. Hence, $V(K_{n-k+1}) \subseteq A$. Since $V(K_{n-k+1}) \subseteq N(u)$, $\forall u \in V(G)$, if $u \in B$, $\deg_A(u) \geq n-k+1 > |B| \geq \deg_B(u)$. Therefore, B must be an empty set, contradicting the assumption that $\langle A, B \rangle$ forms a k-satisfactory partition of G'. Hence, G' is k-unsatisfiable.
□

Since a k-unsatisfiable graph does not contain a partition into global defensive k-alliances, we have the following corollary:

Corollary 39 *For $k \geq 0$, there is no forbidden subgraph characterization of the graphs that do not have a partition into global defensive k-alliances.*

3.5 Satisfiability and Cardinality of Minimum Alliance

In this section, we present results concerning the relationship between the satisfiability of a graph and the cardinality of its minimum alliance. We call a subgraph G' of a graph G to be a k-alliance subgraph of G if $G' = G[A]$ for some k-alliance A in G.

Theorem 40 *For $k \leq 0$, if a graph G with n vertices contains a k-alliance subgraph G' of minimum degree δ' and order $n' < \frac{1}{\Delta-\delta'-k+1}\left((1-k)n + \left\lfloor \frac{\delta^2}{4} \right\rfloor + k\left\lfloor \frac{\delta}{2} \right\rfloor\right)$, then G is k-satisfiable.*

Proof. Let $k \leq 0$ and let $G(V,E)$ be a graph with $|V| = n$. Also, let $G'(V',E')$ be a k-alliance subgraph of G, such that $|V'| = n' < \frac{1}{\Delta-\delta'-k+1}\left((1-k)n + \left\lfloor\frac{\delta^2}{4}\right\rfloor + k\left\lfloor\frac{\delta}{2}\right\rfloor\right)$. Assume to the contrary that G is not k-satisfiable, hence G does not have a k-satisfactory partition. Consider the cutset $S = \langle V', V-V'\rangle$, then $|S| \leq n'(\Delta-\delta')$. Since $V-V'$ is not a k-alliance, there must be some vertex $v \in V-V'$ such that moving v from $V-V'$ to V' decreases the size of the cutset. Let $S_1 = \langle V'+\{v\}, V-V'-\{v\}\rangle = \langle V_1', V-V_1'\rangle$ be the new cutset, then $|S_1| \leq |S|+k-1$. Once again, V_1' is a k-alliance and $\langle V_1', V-V_1'\rangle$ is not a k-satisfactory partition. Therefore there exists a vertex $w \in V-V_1'$, such that moving w from $V-V_1'$ to V_1' will yield a new cutset $S_2 = \langle V_1'+\{w\}, V-V_1'-\{w\}\rangle = \langle V_2', V-V_2'\rangle$ such that $|S_2| \leq |S_1|+k-1$. We continue moving vertices and decreasing the size of the cutset until only $\left\lfloor\frac{\delta}{2}\right\rfloor$ vertices are left in the set that is not a k-alliance. Hence we have $S_{n-n'-\lfloor\frac{\delta}{2}\rfloor} = \left\langle V'_{n-n'-\lfloor\frac{\delta}{2}\rfloor}, V-V'_{n-n'-\lfloor\frac{\delta}{2}\rfloor}\right\rangle$ where $\left|V-V'_{n-n'-\lfloor\frac{\delta}{2}\rfloor}\right| = \left\lfloor\frac{\delta}{2}\right\rfloor$. Hence, $\left\lfloor\frac{\delta}{2}\right\rfloor\left(\left\lfloor\frac{\delta}{2}\right\rfloor+1\right) \leq \left|S_{n-n'-\lfloor\frac{\delta}{2}\rfloor}\right| \leq \left|S_{n-n'-\lfloor\frac{\delta}{2}\rfloor-1}\right| + k - 1 \leq \cdots \leq |S| + (k-1)\left(n-n'-\left\lfloor\frac{\delta}{2}\right\rfloor\right)$, which implies that $|S| \geq \left\lfloor\frac{\delta}{2}\right\rfloor\left(\left\lfloor\frac{\delta}{2}\right\rfloor+1\right)+(1-k)\left(n-n'-\left\lfloor\frac{\delta}{2}\right\rfloor\right)$. But $|S| \leq n'(\Delta-\delta')$, therefore $n'(\Delta-\delta') \geq \left\lfloor\frac{\delta^2}{4}\right\rfloor + (1-k)(n-n') + k\left\lfloor\frac{\delta}{2}\right\rfloor \Rightarrow n' \geq \frac{1}{\Delta-\delta'-k+1}\left((1-k)n + \left\lfloor\frac{\delta^2}{4}\right\rfloor + k\left\lfloor\frac{\delta}{2}\right\rfloor\right)$, hence a contradiction. □

Corollary 41 *For $k \leq 0$, if a graph G with n vertices contains a k-alliance subgraph G' of order $n' < \frac{1}{\left\lfloor\frac{\Delta-k}{2}\right\rfloor-k+1}\left((1-k)n + \left\lfloor\frac{\delta^2}{4}\right\rfloor + k\left\lfloor\frac{\delta}{2}\right\rfloor\right)$, then G is k-satisfiable.*

Proof. The proof is similar as that of Theorem 40. Let $G(V,E)$ be a graph with $|V| = n$, and let $G'(V',E')$ be a k-alliance subgraph of G, such that $|V'| = n' < \frac{1}{\left\lfloor\frac{\Delta-k}{2}\right\rfloor-k+1}$ $\left((1-k)n + \left\lfloor\frac{\delta^2}{4}\right\rfloor + k\left\lfloor\frac{\delta}{2}\right\rfloor\right)$. Assume to the contrary that G is not k-satisfiable. Consider the cutset $S = \langle V', V - V' \rangle$, then $|S| \leq \sum\limits_{v \in V'} \left\lfloor \frac{\deg(v)-k}{2} \right\rfloor \leq \sum\limits_{v \in V'} \left\lfloor \frac{\Delta - k}{2} \right\rfloor = n' \left\lfloor \frac{\Delta-k}{2} \right\rfloor$. But, from the proof of Theorem 40, $|S| \geq \left\lfloor \frac{\delta^2}{4} \right\rfloor + (1-k)(n-n') + k \left\lfloor \frac{\delta}{2} \right\rfloor$. Therefore $n' \left\lfloor \frac{\Delta-k}{2} \right\rfloor \geq \left\lfloor \frac{\delta^2}{4} \right\rfloor + (1-k)(n-n') + k \left\lfloor \frac{\delta}{2} \right\rfloor \Rightarrow n' \geq \frac{1}{\left\lfloor \frac{\Delta-k}{2} \right\rfloor - k + 1} \left((1-k)n + \left\lfloor \frac{\delta^2}{4} \right\rfloor + k \left\lfloor \frac{\delta}{2} \right\rfloor \right)$, hence a contradiction. □

Corollary 42 *If a graph G with $n > \left\lfloor \frac{\Delta^2}{4} \right\rfloor - \left\lfloor \frac{\delta^2}{4} \right\rfloor + \Delta + 1$ vertices contains $\mathbf{K}_{\lceil \frac{\Delta}{2} \rceil + 1}$, then G is 0-satisfiable.*

Proof. Suppose that the graph G is not satisfiable and $V' \subseteq V$ induces a $\mathbf{K}_{\lceil \frac{\Delta}{2} \rceil + 1}$, then V' is a 0-alliance of G. Hence, by Corollary 41, we have;

$$\left\lceil \frac{\Delta}{2} \right\rceil + 1 \geq \frac{1}{\left\lfloor \frac{\Delta}{2} \right\rfloor + 1} \left(n + \left\lfloor \frac{\delta^2}{4} \right\rfloor \right) \Rightarrow \left\lceil \frac{\Delta}{2} \right\rceil \left\lfloor \frac{\Delta}{2} \right\rfloor + \Delta + 1 \geq n + \left\lfloor \frac{\delta^2}{4} \right\rfloor$$

$$\Rightarrow n \leq \left\lfloor \frac{\Delta^2}{4} \right\rfloor - \left\lfloor \frac{\delta^2}{4} \right\rfloor + \Delta + 1$$

Hence a contradiction. □

3.6 Special Cases

In this section, we discuss the satisfiability of some special types of graphs, for example, regular graphs, Eulerian graphs and line graphs.

3.6.1 Satisfiability of Regular Graphs

The following conditions for satisfiability of regular graphs follow from Theorem 40.

Observation 43

(i) *If an r-regular graph G with n vertices contains a defensive (-1)-alliance subgraph of order* $n' < \frac{1}{\lfloor (r+1)/2 \rfloor + 2}\left(2n + \left\lfloor \frac{r^2}{4} \right\rfloor - \left\lfloor \frac{r}{2} \right\rfloor\right)$*, then G is (-1)-satisfiable.*

(ii) *If an r-regular graph G with n vertices contains a defensive 0-alliance subgraph of order* $n' < \frac{1}{\lfloor r/2 \rfloor + 1}\left(n + \left\lfloor \frac{r^2}{4} \right\rfloor\right)$*, then G is 0-satisfiable.*

(iii) *If an r-regular graph G with n vertices contains a q-regular subgraph of order* $n' < \frac{1}{r-q+2}\left(2n + \left\lfloor \frac{r^2}{4} \right\rfloor - \left\lfloor \frac{r}{2} \right\rfloor\right)$*, where* $\left\lfloor \frac{r}{2} \right\rfloor \leq q < r$*, then G is (-1)-satisfiable.*

(iv) *If an r-regular graph G with n vertices contains a q-regular subgraph of order* $n' < \frac{1}{r-q+1}\left(n + \left\lfloor \frac{r^2}{4} \right\rfloor\right)$*, where* $\left\lceil \frac{r}{2} \right\rceil \leq q < r$*, then G is 0-satisfiable.*

(v) *If an r-regular graph G with $n > r+1$ vertices contains* $\mathbf{K}_{\left\lfloor \frac{r}{2} \right\rfloor + 1}$*, then G is (-1)-satisfiable.*

(vi) *If an r-regular graph G with $n > r+1$ vertices contains* $\mathbf{K}_{\left\lceil \frac{r}{2} \right\rceil + 1}$*, then G is 0-satisfiable.*

We proceed by showing that (3,4)-regular graphs, with a few exceptions, are 0-satisfiable. A *(3,4)-regular graph* is a graph G with $3 \leq \delta(G) \leq \Delta(G) \leq 4$.

Lemma 44 *A set $S \subseteq V(G)$, where G is a (3,4)-regular graph, is a minimal 0-alliance if and only if S induces a cycle of G.*

Proof. Suppose there exists a minimal defensive 0-alliance A in a (3,4)-regular graph G and let G' be the subgraph induced by A. First assume that G' is acyclic. Since A is minimal, G' must be connected and hence is a tree. Consider any leaf v of this tree. Since the degree of v in G is either 3 or 4, v must be connected to at least 2 vertices outside A. Hence, A is not a defensive 0-alliance, contradicting our initial assumption. Therefore G' must contain a cycle and $\delta(G') \geq 2$.

Now assume that G' has more than one cycle. But then each cycle is also a defensive 0-alliance, contradicting that A is minimal. Hence, G' is a cycle.

For the converse, let A be the set of vertices of any induced cycle of a (3,4)-regular graph. Clearly, A is a defensive 0-alliance. Assume, to the contrary, that A is not minimal. Then a proper subset of A must be a defensive 0-alliance. But every proper subset of A induces a forest and has at least two vertices of degree less than 2 that are not satisfied, a contradiction. □

Corollary 45 *A (3,4)-regular graph is 0-satisfiable if and only if it has at least two vertex disjoint cycles.*

We now characterize the (3,4)-regular graphs that have vertex disjoint cycles. Let Q be the set of graphs that have $n-3$ independent degree 3 vertices, where n is the number of vertices. A wheel W_n is a cycle on $n-1$ vertices plus a single vertex adjacent to all vertices of the cycle.

Lemma 46 *If $\delta(G) \geq 3$, then G has two disjoint cycles if and only if $n \geq 6$, G is not a wheel, and G is not in Q.*

Proof. If G has less than 6 vertices, then it cannot have vertex disjoint cycles. Since every cycle in a wheel contains a common vertex or has $n-1$ vertices, it cannot have vertex disjoint cycles. Suppose that $G \in Q$ and let A be the set of $n-3$ independent degree 3 vertices. Then every cycle in G must contain at least 2 vertices from $V-A$, hence G does not have vertex disjoint cycles.

We prove the converse by induction on the number of vertices. By case analysis, it can be seen that there are at least two vertex disjoint cycles in every graph G with $\delta(G) \geq 3$ and $n = 6$ when G is not a wheel and is not in Q. Assume the statement is true for all graphs with order $n \leq k$ for arbitrary $k \geq 6$.

Consider a graph G with $\delta(G) \geq 3$ and $n = k+1$, G is not a wheel, and is not in Q. We pick a vertex v in G such that i) $\deg(v) = \delta(G)$, ii) Among all vertices of minimum degree, v maximizes the number of edges induced by $N(v) \cup \{v\}$. Consider the graph $G-v$. If $G-v$ is not a wheel and is not in Q, and $\delta(G-v) \geq 3$ then by induction hypothesis $G-v$ has at least two vertex disjoint cycles, hence G has at least two vertex disjoint cycles.

Assume that $\delta(G-v) < 3$, then $\deg_G(v) = 3$. Let $N(v) = \{v_1, v_2, v_3\}$, where the degree of at least one of v_i is 2 in graph $G-v$. Assume without loss of generality that

$\deg_{G-v}(v_1) = 2$. Let G_1 be the graph obtained by adding edges in $G - v$ between the vertices of $N(v)$, such that $\delta(G_1) \geq 3$. Let $E' = E(G_1) - E(G)$, where $1 \leq |E'| \leq 2$.

Suppose that we cannot construct G_1 by adding edges because the vertices were already adjacent, then we have a triangle say vv_1v_2v in graph G such that v and v_1 are adjacent to exactly one vertex in $V' = V - \{v, v_1, v_2\}$. If there is any cycle in V', then the graph G has 2 vertex disjoint cycles. Now assume that the graph $G[V']$ is acyclic. Since $\delta(G) = 3$, every vertex in $G[V']$ with degree less than 2 must be connected to at least 2 vertices of the triangle, hence, $G[V']$ must be a path with each internal vertex having an edge to v_2. Therefore G is a wheel, contradicting our assumption.

Let $G_1 \in Q$ and let V_1 be the set of $k - 3$ independent vertices and $V_2 = V - V_1$. If $V_1 \cap N(v) = \emptyset$ then $G \in Q$, whereas if $V_2 \cap N(v) = \emptyset$ then $|E'| = 0$, a contradiction. Since $\forall w \in V_2$, $\deg_{V_1}(w) \geq 3$, therefore $\forall e \in E'$, $e = xy \Rightarrow x \in V_1 \wedge y \in V_2$. Also, since $\forall w \in V_1$, $\deg(w) = 3$, every vertex in V_1 is end vertex of at most one edge in E'. Now consider two cases. Case 1. $|E'| = 1$: Let $e = v_1v_2 \in E'$ such that $v_1 \in V_1$ and $v_2 \in V_2$, then there are two vertex disjoint cycles in G, a triangle T (where $T = vv_1v_3v$, if $v_3 \in V_2$ and $T = vv_2v_3v$, if $v_3 \in V_1$) and $wxyzw$ where $w, y \in V_1 - T$ and $x, z \in V_2 - T$. Case 2. $|E'| = 2$: Let $e_1, e_2 \in E'$ where $e_1 = v_1v_2$ and $e_2 = v_3v_2$. Then the only possibility is that $|V_1| \geq 4$, $v_1, v_3 \in V_1$ and $v_2 \in V_2$. Again there are two vertex disjoint cycles in G, vv_1pv_3v (where $p \in V_2 - \{v_2\}$) and $wxyzw$ where $w, y \in V_1 - N(v)$ and $x, z \in V_2 - \{p\}$.

Let G_1 be a wheel such that $X = \{x_1, x_2, \ldots, x_{k-1}\}$ forms a cycle C, and y is a vertex adjacent to every vertex of X. Since $\forall x_i \in X - N(v)$, $\deg(x_i) = 3$ and $N(x_i) \cup \{x_i\}$ induces

5 edges in G, by choice of v, $N(v) \cup \{v\}$ must induce at least 5 edges in G. But this is possible only if $\exists x_j, x_{j+1} \in N(v)$. If $x_j x_{j+1} \in E(G)$ then there are two vertex disjoint cycles $v x_j x_{j+1} v$ and $y x_{j+2} x_{j+3} y$ in G_1. Otherwise $vy \in E(G)$ and hence G is a wheel, a contradiction.

We may now assume that G_1 is not a wheel and is not in Q. Hence by induction hypothesis, G_1 has two vertex disjoint cycles. If these cycles do not include the edges in E', then G has two vertex disjoint cycles. If any of these cycles in G_1 include a path (assume $v_1 v_2$) consisting of edges in E', then it can be extended in G by replacing the path $v_1 v_2$ by edges $v_1 v$ and $v v_2$. Hence, G has two vertex disjoint cycles.

Now assume that $G - v$ is a wheel such that $X = \{x_1, x_2, \ldots, x_{k-1}\}$ forms a cycle C, and y is a vertex adjacent to every vertex of X. Then, in G, v must be adjacent to at least two vertices $x_i, x_j \in X$. Let x_m, x_{m+1} be two adjacent vertices in one of the $x_i - x_j$ paths in C Then $y x_m x_{m+1} y$ and $v x_i - x_j v$ forms two vertex disjoint cycles in G.

Finally, assume that $G - v \in Q$. Let $A = \{a_1, a_2, \ldots, a_{k-3}\}$ be the set of $k-3$ independent degree 3 vertices and $B = \{b_1, b_2, b_3\}$ be the remaining 3 vertices. Then v must be adjacent to at least one vertex $a_i \in A$, otherwise $G \in Q$. If v is connected to any vertex in B, say b_1, then G has at least two vertex disjoint cycles, $v a_i b_1 v$ and the other formed by b_2, b_3 and any two vertices in A other than a_i. If v is not connected to any vertex in B, let $a_i, a_j \in N(v)$, then again there are two vertex disjoint cycles, $v a_i b_1 a_j v$ and $b_2 a_p b_3 a_q b_2$ where a_p and a_q are vertices in A other than a_i and a_j. The vertices a_p and a_q always exist for all $k > 6$.

When $k = 6$ then either there are vertex disjoint cycles, $va_ib_1a_jv$ and $a_pb_2b_3a_p$ or $G \in Q$, contradicting the hypothesis. □

Theorem 47 *A (3,4)-regular graph G is 0-satisfiable if and only if $n \geq 6$, G is not a wheel and G is not in Q.*

Corollary 48

(i) *Every (3,4)-regular graph of order $n \geq 8$ is 0-satisfiable.*

(ii) *Every 4-regular graph except $\mathbf{K}_5$ is 0-satisfiable.*

(iii) *Every 3-regular graph except $\mathbf{K}_{3,3}$ and $\mathbf{K}_4$ is 0-satisfiable.*

(iv) *Every (3,4)-regular graph except $\mathbf{K}_5$ is (-1)-satisfiable.*

We believe, but have been unable to prove that the following generalization of Corollary 48(ii) is true.

Conjecture 49 *Every finite $2k$-regular graph with more than $2k+1$ vertices is 0-satisfiable.*

We prove next a weaker result that all triangle free Eulerian graphs are 0-satisfiable. In addition, we show that all graphs that do not have a triangle of even vertices are (-1)-satisfiable.

3.6.2 Satisfiability of Odd Graphs and Triangle free Eulerian Graphs

We define a set A to be *degenerate* if $\forall S \subseteq A$, $\exists v \in S$ such that $\deg_S(v) \leq \left\lfloor \frac{\deg(v)}{2} \right\rfloor$. It is *strong degenerate* if the inequality is strong. If a set A is (strong) degenerate, then $\forall S \subseteq A$, S is also (strong) degenerate. If A is not strong degenerate then $\exists S \subseteq A$, such that $\forall v \in S$, $\deg_S(v) \geq \left\lfloor \frac{\deg(v)}{2} \right\rfloor$, i.e., A contains a defensive (-1)-alliance. Similarly, if A is not degenerate then $\exists S \subseteq A$, such that for all $v \in S$, $\deg_S(v) > \left\lfloor \frac{\deg(v)}{2} \right\rfloor$.

Theorem 50 *A graph that does not contain any triangle of even vertices is (-1)-satisfiable.*

Proof. The proof follows a similar reasoning as in [Kan98]. Let G be a graph that does not contain any triangle of even vertices. Assume to the contrary that G is (-1)-unsatisfiable. From Observation 29, $\delta(G) \geq 2$. Consider a partition $\langle A, B \rangle$ of $V(G)$ such that A is degenerate containing a defensive (-1)-alliance, say T. Since every minimal defensive (-1)-alliance is degenerate, such a partition always exists. Let the partition $\langle A, B \rangle$ be such that the edge cutset $S = \langle A, B \rangle$ is minimum among all such partitions. Further assume that A is minimum subject to these properties. Since A contains a defensive (-1)-alliance, and since $\delta(G) \geq 2$, $|A| \geq 2$. Since A is degenerate, there is a vertex $v \in A$ such that $\deg_A(v) \leq \left\lfloor \frac{\deg(v)}{2} \right\rfloor$, hence $|B| \geq \left\lfloor \frac{\delta(G)}{2} \right\rfloor \geq 1$.

Suppose $|B| = 1$, and let $q \in B$, then $\exists r \in A$ such that $\deg(r) = 2$ and $qr \in E(G)$. Consider the partition $\langle A - \{r\}, B \cup \{r\} \rangle$. By definition, $A - \{r\}$ is degenerate and the size of the new edge cutset is equal to $|S|$. By minimality of A, the only alternative is that $A - \{r\}$ does not contain any defensive (-1)-alliance, that is $\exists s \in A - \{r\}$ such that

$\deg_{A-\{r\}}(s) < \left\lfloor \frac{\deg(s)}{2} \right\rfloor$. This is possible only if $\deg_{A-\{r\}}(s) = 0$ and $\deg(s) = 2$, which implies that $\{r, s\}$ is a minimal defensive (-1)-alliance in G. Consider now the partition $\langle \{r, s\}, V - \{r, s\} \rangle$. Since $\{r, s\}$ is a minimal defensive -1-alliance, it is also degenerate. Moreover, the size of the cutset $T = \langle \{r, s\}, V - \{r, s\} \rangle$ is at most the size of the cutset $S = \langle A, B \rangle$. Since A was minimum such set, $|A| = 2$, i.e., G is a triangle, a contradiction. Hence, $|B| \geq 2$.

Recall that if B is not strong degenerate then it contains a (-1)-alliance, say C. But then there are two vertex-disjoint (-1)-alliances C and T in G, a contradiction. So we may assume that B is strong degenerate, i.e., $\exists x \in B$ such that $\deg_B(x) < \left\lfloor \frac{\deg(x)}{2} \right\rfloor$.

Let $D = \left\{ v \in A | \deg_A(v) \leq \left\lfloor \frac{\deg(v)}{2} \right\rfloor \right\}$ and $R = \left\{ w \in B | \deg_B(w) < \left\lfloor \frac{\deg(w)}{2} \right\rfloor \right\}$. Since A is degenerate and B is strong degenerate, $D \neq \emptyset$ and $R \neq \emptyset$.

We claim that for any (-1)-alliance $T' \subseteq A$, $D \subseteq T'$. Suppose not. Then there exists a vertex $v \in A - T'$ such that $\deg_A(v) \leq \left\lfloor \frac{\deg(v)}{2} \right\rfloor$. Hence the size of cutset $S' = \langle A - \{v\}, B \cup \{v\} \rangle$ is at most $|S|$. By definition $A - \{v\}$ is degenerate and since $T' \subseteq A - \{v\}$, $A - \{v\}$ contains a (-1)-alliance, which is a contradiction since A is a minimal such set. Hence $D \subseteq T'$, as claimed. This also implies that $\forall v \in D$, $\deg_A(v) = \left\lfloor \frac{\deg(v)}{2} \right\rfloor$. Hence A is a (-1)-alliance. Furthermore, $|D| > 1$ and for all $v \in D$, $N(v) \cap D \neq \emptyset$.

Now we claim that for all $x \in R$, $D \subseteq N(x)$. Suppose not. Consider the partition $\langle A \cup \{x\}, B - \{x\} \rangle$, the cutset $S' = \langle A \cup \{x\}, B - \{x\} \rangle$ is strictly smaller than $|S|$. Hence $A \cup \{x\}$ can not be degenerate, i.e., there exists a (-1)-alliance $T' \subseteq A$ such that $\forall v \in$

$T' \cup \{x\}$, $\deg_{T' \cup \{x\}}(v) > \left\lfloor \frac{\deg(v)}{2} \right\rfloor$. Since $D \subseteq T'$, and $\forall v \in D$, $\deg_A(v) = \left\lfloor \frac{\deg(v)}{2} \right\rfloor$, $D \subseteq N(x)$.

Let $x \in R$ and $y \in D$ and consider the partition $\langle (A \cup \{x\}) - \{y\}, (B \cup \{y\}) - \{x\} \rangle$. The size of cutset $S'' = \langle A - \{y\}, B \cup \{y\} \rangle$ is less than or equal to $|S|$. Suppose $|S''| < |S|$. We know that $A' = (A \cup \{x\}) - \{y\}$ contains a defensive (-1)-alliance $T' = (T \cup \{x\}) - \{y\}$. The only alternative is that A' is not degenerate, i.e., $\forall v \in T'$, $\deg_{T'}(v) > \left\lfloor \frac{\deg(v)}{2} \right\rfloor$. Which is only possible if $|D| = 1$ or $N(y) \cap D = \emptyset$, a contradiction. Hence, $|S''| = |S|$, which implies that both x and y have even degrees. Thus, all vertices in $R \cup D$ have even degrees. Since $|D| > 1$ and $\forall v \in D$, $N(v) \cap D \neq \emptyset$, and since $\forall x \in R$, $D \subseteq N(x)$, the graph $G[D \cup R]$ contains a triangle. This contradicts that G does not contain a triangle of even vertices. Hence, our initial assumption that G is (-1)-unsatisfiable must be incorrect. □

Corollary 51

(i) *Every odd graph is* (-1)*-satisfiable.*

(ii) *Every triangle free Eulerian graph is* 0*-satisfiable.*

(iii) *Every triangle free* $2k$*-regular graph is* 0*-satisfiable.*

3.6.3 Satisfiability of Line Graphs

A line graph $L(G)$ of a graph G is obtained by associating a vertex with each edge of the graph and connecting two vertices with an edge if and only if the corresponding edges of G meet at one or both endpoints. In this section, we characterize the graphs whose line graphs are satisfiable.

Theorem 52 *If a graph G has at least $\Delta^2 - \delta^2 + 2\delta$ edges, then the line graph $L(G)$ is 0-satisfiable.*

Proof. Let G be a graph with n vertices. The line graph $L(G)$ of G is a graph with $m = E(G)$ vertices such that $\frac{n\delta}{2} \leq m \leq \frac{n\Delta}{2}$. Let maximum and minimum degrees of $L(G)$ be Δ' and δ' respectively, then, $\Delta' \leq 2\Delta - 2$ and $2\delta - 2 \leq \delta' \leq \Delta + \delta - 2$. The edges incident to a vertex v in G form a clique in $L(G)$, therefore $L(G)$ contains $\mathbf{K}_\Delta$. Since $\Delta \geq \frac{\Delta'}{2} + 1$, Corollary 42 implies $L(G)$ is 0-satisfiable whenever

$$m > \left\lfloor \frac{\Delta'^2}{4} \right\rfloor - \left\lfloor \frac{\delta'^2}{4} \right\rfloor + \Delta' + 1$$

Since $\Delta' \leq 2\Delta - 2$ and $\delta' \geq 2\delta - 2$, we have,

$$\left\lfloor \frac{\Delta'^2}{4} \right\rfloor - \left\lfloor \frac{\delta'^2}{4} \right\rfloor + \Delta' + 1 < \left\lfloor \frac{(2\Delta-2)^2}{4} \right\rfloor - \left\lfloor \frac{(2\delta-2)^2}{4} \right\rfloor + (2\Delta - 2) + 2$$

$$= \Delta^2 - \delta^2 + 2\delta.$$

Hence, if $m \geq \Delta^2 - \delta^2 + 2\delta$ then $L(G)$ is 0-satisfiable. □

Corollary 53 *If a graph G is r-regular with $n > 3$ vertices then the Line graph $L(G)$ of G is 0-satisfiable.*

A vertex v is '*degree k-dominant*' if $\deg(v) \geq \deg(w) + k$, $\forall w \in N(v)$.

Let $E_U = \{e \in E | e = uv, \ \forall u \in U\}$, $\forall U \subseteq V(G)$ and $f : 2^{E(G)} \to 2^{V(L(G))}$ such that $f(E')$ is the set of vertices in $L(G)$ corresponding to the edges in $E' \subseteq E(G)$. For simplicity of notation, we will denote single element subsets $\{e\}$ by the elements themselves, i.e., e. Note that f is one to one and onto and $|E_u| = |f(E_u)| = \deg(u)$. $E_u \cap E_v = uv$ if $uv \in E(G)$, otherwise $E_u \cap E_v = \emptyset$. Also $f(E_u) \cap f(E_v) = f(E_u \cap E_v)$.

Lemma 54 *If v is degree k-dominant vertex of a graph G then $f(E_v)$ is a defensive k-alliance in $L(G)$.*

Proof. Let v be a degree k-dominant vertex of a graph G then $f(E_v)$ forms a clique $\mathbf{K}_{\deg(v)}$ in $L(G)$. Suppose $\exists w = f(uv) \in f(E_v)$ that is not k-satisfied in $f(E_v)$ then $\deg(v) - 1 = |N(w) \cap f(E_v)| < |N(w) \cap f(E_u)| + k = \deg(u) - 1 + k$. This implies that $\deg(v) < \deg(u) + k$, which is not possible since v is degree k-dominant and $u \in N(v)$. Hence $f(E_v)$ is a defensive k-alliance in $L(G)$. $\square$

Corollary 55 *For $k \leq 0$, if a graph G has two non adjacent degree k-dominant vertices u and v then the Line graph $L(G)$ of G is k-satisfiable.*

Proof. Let $u, v \in V(G)$, such that both u and v are degree k-dominant. Both $f(E_u)$ and $f(E_v)$ form cliques $\mathbf{K}_{\deg(u)}$ and $\mathbf{K}_{\deg(v)}$ respectively and by Lemma 54, are defensive k-alliances in $L(G)$. Since $uv \notin E(G)$, $f(E_u) \cap f(E_v) = \emptyset$, i.e., $f(E_u)$ and $f(E_v)$ are disjoint defensive k-alliances in $L(G)$, hence $L(G)$ is k-satisfiable. $\square$

Theorem 56 *If a graph G has a degree 0-dominant vertex not adjacent to any degree 2 vertex then its line graph $L(G)$ is 0-satisfiable if and only if G is not a star,* $\mathbf{K}_{1,n-1}$.

Proof. Let v be a degree 0-dominant vertex of a graph G and $\forall w \in N(v), \deg(w) \neq 2$. Then, by Lemma 54, $f(E_v)$ is a defensive 0-alliance in $L(G)$. Let $V_2 = V(L(G)) - f(E_v)$. Case 1: $V_2 = \emptyset$. The set $V_2 = \emptyset$ if and only if G is a star. Since complete graphs are not 0-satisfiable, $L(G)$ is not 0-satisfiable if G is a star. Case 2: $V_2 \neq \emptyset$. Then every vertex in V_2 is adjacent to at most 2 vertices of $f(E_v)$. Also, since $N(v)$ has no degree 2 vertex, $\forall u \in V_2, |N(u) \cap V_2| \geq |N(u) \cap f(E_v)|$. Thus V_2 is also a defensive 0-alliance in $L(G)$. Hence, $L(G)$ is 0-satisfiable. □

Theorem 57 *Assume $L(G)$ is not 0-satisfiable and let u and v be the largest and second largest degree vertices in G respectively* ($\deg(v) \leq \Delta(G)$)*, then for every integer r in the interval* $[2, \deg(v)]$ *,* $\exists w \in N(u)$ *such that* $\deg(w) = r$.

Proof. From Theorems 56 and Corollary 55, we know that the statement is true for $r = 2$ and $r = \deg(v)$. Assume to the contrary that $\exists r$, $2 < r < \deg(v)$, such that $\forall w \in N(u)$, $\deg(w) \neq r$. By Lemma 54, we know that $f(E_u)$ is a defensive 0-alliance in $L(G)$. Let $V' = \{x \in V(G) - N[u] \,|\, \deg(x) \geq r\} \cup \{x \in N(u) \,|\, \deg(x) \geq r+1\}$ and consider the set $C = f(E_{V'}) - f(E_u)$. Since $v \in V'$, $|V'| \neq \emptyset$. Also, since $L(G)$ is not 0-satisfiable , C is not a defensive 0-alliance otherwise there are two vertex disjoint defensive 0-alliances in $L(G)$. Hence, there must exist $w = f(xy) \in C$, such that $\deg_C(w) < \deg_{V(L(G))-C}(w)$. Assume without any loss of generality that $x \in V'$ and consider two exhaustive cases.

Case 1: $x \notin N(u)$. By the definition of V', we know that $\deg(x) \geq r$. Since $f(E_x) \subseteq C$ and $w \in f(E_x)$, we must have $|f(E_y) - C| > |N(w) \cap C| \geq \deg(x) - 1$. This implies that $|f(E_y) - C| \geq r$. Since $w \in f(E_y) \cap C$, $|f(E_y)| = \deg(y) \geq r + 1$. But this means that $y \in V'$ and hence $|f(E_y) - C| \leq 1$, a contradiction.

Case 2: $x \in N(u)$. By the definition of V', we know that $\deg(x) \geq r + 1$. Since $f(E_x) - f(ux) \subseteq C$, $w \in f(E_x)$, and $f(ux) \in N(w) - C$, we must have $|f(E_y) - C| \geq |N(w) \cap C| \geq \deg(x) - 2$. This implies that $|f(E_y) - C| \geq r - 1$. Since $w \in f(E_y) \cap C$, $|f(E_y)| = \deg(y) \geq r$. If $y \notin N(u)$ or $\deg(y) \geq r + 1$ then $y \in V'$ and hence $|f(E_y) - C| \leq 1$, which is a contradiction. Otherwise $y = N(u)$ and $\deg(y) = r$, which is again contrary to the initial assumption that $\forall w \in N(u)$, $\deg(w) \neq r$. $\square$

Let J be the set of graphs of n vertices, such that every graph G in J satisfies the following properties:

(i) n is even

(ii) There is a vertex u in $V(G)$ of degree $n - 1$.

(iii) $\forall w \in V - \{u\}$, $\deg(w) \leq 2$.

(iv) The number of degree 1 vertices is greater than $n/2$.

Theorem 58 *A line graph $L(G)$ is (-1)-satisfiable if and only if G is not in J and G is not a triangle.*

Proof. The line graph of a triangle is also a triangle, which is not (-1)-satisfiable. Let G be a graph of order n in J then the line graph $L(G)$ of G contains at most $5n/4 - 2$ vertices and a clique K_{n-1} of order $n-1$. Let $X = V(L(G)) - V(K_{n-1})$, then X is an independent set of degree 2 vertices, such that, $|X| \leq n/4 - 1$ and $\forall\{a, b\} \subset X$, $N(a) \cap N(b) = \emptyset$. Also let $Y = \{x \in V(K_{n-1}) | \deg(x) = n-2\}$ and $Z = \{x \in V(K_{n-1}) | \deg(x) = n-1\}$. By the above argument, $V(K_{n-1}) = Y \cup Z$, and $|Y| \geq n - 1 - 2(n/4 - 1) = n/2 + 1$. Assume to the contrary that $L(G)$ has a (-1)-satisfactory partition, A, B. Consider $x \in Y$ and without loss of generality, assume that $x \in A$. Since $\deg(x) = n-2$ and $N(x) = Y \cup Z$, $|A \cap (Y \cup Z)| \geq n/2$. But then no vertex in Y is (-1)-satisfied in B, and hence, $Y \subseteq A$. Thus, $|A| \geq n/2 + 1$. Since, for all vertices $x \in Z$, $\deg(x) = n-1$ and $Y \subseteq N(x)$, x cannot be (-1)-satisfied in B. Hence $Z \subseteq A$. But then the vertices in X cannot be (-1)-satisfied in B, and B must be empty, a contradiction.

To prove the sufficiency part of the theorem, we first show that if G is not (-1)-satisfiable then G has at most one vertex of degree greater than 2. Let u and v be the largest and second largest degree vertices in a (-1)-unsatisfiable graph G respectively, $(\deg(v) \leq \Delta(G))$. Assume to the contrary that $\deg(v) > 2$. By Corollary 55, $uv \in E(G)$. By Lemma 54, we know that $f(E_u)$ is a defensive (-1)-alliance in $L(G)$. Let $V' = \{x \in V(G) - N[u] \,|\, \deg(x) \geq 2\} \cup \{x \in N(u) \,|\, \deg(x) \geq 3\}$ and consider the set $C = f(E_{V'}) - f(E_u)$. Since $v \in V'$, $|V'| \neq \emptyset$. Also, since $L(G)$ is not (-1)-satisfiable , C is not a defensive (-1)-alliance. Hence, there must exist $w = f(xy) \in C$, such that $\deg_C(w) < \deg_{V(L(G))-C}(w) - 1$. Assume without any loss of generality that $x \in V'$ and consider two exhaustive cases. Case 1: $x \notin N(u)$. By the definition of V', we know that $\deg(x) \geq 2$. Since $f(E_x) \subseteq C$ and $w \in f(E_x)$, we must have $|f(E_y) - C| - 1 > |N(w) \cap C| \geq \deg(x) - 1$. This implies that $|f(E_y) - C| \geq 3$. Since $w \in f(E_y) \cap C$, $|f(E_y)| = \deg(y) \geq 4$. But this means that $y \in V'$ and hence $|f(E_y) - C| \leq 1$, a contradiction. Case 2: $x \in N(u)$. By the definition of V', we know that $\deg(x) \geq 3$. Since $f(E_x) - f(ux) \subseteq C$, $w \in f(E_x)$, and $f(ux) \in N(w) - C$, we must have $|f(E_y) - C| > |N(w) \cap C| \geq \deg(x) - 2$. This implies that $|f(E_y) - C| \geq 2$. Since $w \in f(E_y) \cap C$, $|f(E_y)| = \deg(y) \geq 3$. Hence, $y \in V'$ and $|f(E_y) - C| \leq 1$, a contradiction. Since all cases lead to contradiction, we must conclude that for all vertices $w \in V - \{u\}$, $\deg(w) \leq 2$.

Suppose now that $V - N[u] \neq \emptyset$, then every vertex $w \in V - N[u]$ is degree (-1)-dominant. But then u and w are two non adjacent degree (-1)-dominant vertices in G, which is contrary to $L(G)$ being (-1)-unsatisfiable. Hence $deg(u) = n - 1$.

It is easy to see that if $n = |V|$ is odd or if the number of degree 1 vertices in G is less or equal to $n/2$ then $L(G)$ is (-1)-satisfiable. Hence, G is either in J or G is a triangle. □

3.7 Computational Complexity

Stiebitz [Sti96] showed that the problem of partitioning a graph into defensive k alliances is polynomial when $k < -1$. The problem is also polynomial for $k = -1$ when restricted to odd graphs. The problem is NP-Complete for all $k \geq 0$. Here, we show that the problem PARTITION INTO GLOBAL DEFENSIVE ALLIANCES (PGDA) is NP-Complete by giving a polynomial transformation from NAE3SAT problem, which is defined as follows:

NOT ALL EQUAL 3SAT (NAE3SAT)

Input: A set $U = \{u_1, u_2, \ldots, u_n\}$ of variables and a collection $C = \{C_1, C_2, \ldots, C_m\}$ of clauses over U, where each clause contains exactly three literals (variables or their complements), with no literal appearing more than once in any given clause.

Question: Is there a truth assignment that makes one or two (but not all three) literals true in each clause?

We may assume that each literal appears in at least one of the clauses, otherwise, for each literal u_j that does not appear in any of the clauses, we can add another variable y and two clauses $C'_1 = \{u_j, \overline{u_j}, y\}$ and $C'_2 = \{u_j, \overline{u_j}, \overline{y}\}$. These two clauses are satisfied by any truth assignment and do not affect the truth assignment of the original problem.

Theorem 59 *Given a graph G, the problem of deciding whether the graph G has a partition into global defensive alliances is NP-Complete.*

Proof. Given an instance of NAE3SAT with n variables and m clauses, we transform it into an instance of PGDA by constructing a graph $G = (V, E)$ as follows:

For a literal $u \in U \cup \overline{U}$, let $C(u)$ be the set of clauses that contains u. Let $V = Q \cup X \cup R \cup T$, where $Q = \{q(u), u \in (U \cup \overline{U})\}$, $X = \{x_i, 1 \leq i \leq n\}$, $R = \left(\bigcup_{u \in (U \cup \overline{U})} R(u)\right)$, and $T = \left(\bigcup_{1 \leq j \leq m} T_j\right)$. For all $u \in U$, $R(u) = \{r_i(u), 1 \leq i \leq |C(u)| + 2\}$, and for all $\overline{u} \in \overline{U}$, $R(\overline{u}) = \{r_i(\overline{u}), 1 \leq i \leq |C(\overline{u})|\}$. Also, for all j, $1 \leq j \leq m$, $T_j = \{t_j(a), t_j(b), t_j(c) | C_j = \{a, b, c\}\}$. For each literal $u \in U \cup \overline{U}$, we create a star, $S(u)$, where $V(S(u)) = \{q(u)\} \cup R(u)$ and the vertex $q(u)$ forms the center of the star. For each $x_i \in X$, we add edges $x_i q(u_i)$ and $x_i q(\overline{u_i})$ in graph G. For each clause $C_j \in C$, we setup a triangle T_j in V and for each vertex $t_j(u) \in T_j$, add an edge $q(u)t_j(u)$ in graph G.

The order of the constructed graph, $|V| = 4n + 6m$ and the size of the graph, $|E| = 3n + 9m$, which is polynomially related to the size of the NAE3SAT problem.

We now claim that the constructed graph G has a partition into global defensive alliances if and only if the given instance of NAE3SAT has a satisfying truth assignment. The proof of the claim is as follows:

$\Longrightarrow$ Suppose that the given instance of NAE3SAT has a satisfying truth assignment $f : U \longrightarrow \{0,1\}$. We define a partition of the vertex set $V = A \cup B$ as follows: $A = \bigcup_{u \in U(1)} \cup \{s_i | f(u_i) = 1\} \cup \{\overline{s_i} | f(u_i) = 0\}$ and $B = V - A$. We now show that $\forall v \in V$, $N(v) \cap A \neq \emptyset$ and $N(v) \cap B \neq \emptyset$, i.e., A is an 'open neighborhood' free cover. We consider three cases. Case 1: $v \in R$. Since f is a satisfying assignment, every clause C_i contains a literal that is assigned the value 1 and a literal that is assigned the value 0. Hence, for all $v \in R$, v is adjacent to at least one vertex in the set A and at least one vertex in the set B. Case 2: $v \in S \cup \overline{S}$. By assumption, each literal appears in at least one of the clauses. Hence, each vertex in set $S \cup \overline{S}$ is adjacent to at least one vertex in $R \subseteq A$. Also, by construction, each vertex in set $S \cup \overline{S}$ is adjacent to one vertex in $T \subseteq B$. Case 3: $v \in T$. By construction, each $v \in T$ is adjacent to a vertex $s_i \in S$ and $\overline{s_i} \in \overline{S}$ and thus has a neighbor in both sets A and B.

$\Longleftarrow$ Suppose now that the constructed graph G has an 'open neighborhood' free cover A, and let $B = V - A$. We define a truth assignment $f : U \longrightarrow \{0, 1\}$, such that $f(u_i) = 1$ if and only if $s_i \in A$. Since each vertex $t_i \in T$ is adjacent to only two vertices, $s_i \in S$ and $\overline{s_i} \in \overline{S}$, exactly one of these vertices must be in set A. Thus, for each literal u_i, $f(u_i) = 1$ if and only if $f(\overline{u_i}) = 0$, i.e., f is a legal assignment. Also, each vertex $r_i \in R$ has at least one vertex in A and one vertex in B and hence each clause C_i has at least one true literal and at least one false literal. Thus, f is a satisfying assignment. □

CHAPTER 4

ALLIANCE FREE AND ALLIANCE COVER SETS

4.1 Introduction

In this chapter, we introduce the concept of alliance-free and alliance cover sets, where an alliance free set (for some type of alliance) is a set that does not contain any alliance (of that type), while an alliance cover set (for some type of alliance) is a set that contains at least one member (vertex) of each alliance (of that type). In particular, we consider the alliance free and cover sets in the context of $k-$defensive and $k-$offensive alliances as defined in Chapter 2.

Consider a graph $G = (V, E)$ without loops or multiple edges. Recall that a vertex v in set $A \subseteq V$ is said to be k-satisfied with respect to A if $\deg_A(v) \geq \deg_{V-A}(v) + k$, where $\deg_A(v) = |N(v) \cap A| = |N_A(v)| = \deg(v) - \deg_{V-A}(v)$. A set A is a *defensive k-alliance* if all vertices in A are $k-$satisfied with respect to A, where $-\delta < k \leq \delta$. Similarly, a set $A \subseteq V$ is an offensive $k-$alliance if $\forall v \in \partial A, \deg_A(v) \geq \deg_{V-A}(v) + k$, where $-\delta + 2 < k \leq \delta$.

A set $X \subseteq V$ is *defensive $k-$alliance free* ($k-$daf) if for all defensive $k-$alliances A, $A - X \neq \emptyset$, i.e., X does not contain any defensive $k-$alliance as a subset. A defensive $k-$alliance free set X is maximal if $\forall v \notin X$, $\exists S \subseteq X$ such that $S \cup \{v\}$ is a defensive $k-$alliance. A maximum $k-$daf set is a maximal $k-$daf set of largest cardinality. Let $\phi_k(G)$ be the cardinality of a maximum $k-$daf set of graph G. For simplicity of notation, we will refer to a maximum $k-$daf set of G as a $\phi_k(G)$-set. If a graph G does not have a defensive $k-$alliance (for some k), we say that $\phi_k(G)= |V(G)| = n$, for example, $\phi_k(P_n)= n$, $\forall k > 1$. Since $\forall k_1 \geq k_2$, a defensive k_2-alliance free set is also defensive k_1-alliance free, we have $\phi_{k_1}(G) \geq \phi_{k_2}(G)$ if and only if $k_1 \geq k_2$.

We define a set $Y \subseteq V$ to be a *defensive $k-$alliance cover* ($k-$dac) if for all defensive $k-$alliances A, $A \cap Y \neq \emptyset$, i.e., Y contains at least one vertex from each defensive $k-$alliance of G. A $k-$dac set Y is minimal if no proper subset of Y is a defensive $k-$alliance cover. A minimum $k-$dac set is a minimal cover of smallest cardinality. Let $\zeta_k(G)$ be the cardinality of a minimum $k-$dac set of graph G. Once again, we will refer to a minimum $k-$dac set of G as a $\zeta_k(G)$-set. When G does not have a defensive $k-$alliance (for some k), we say that $\zeta_k(G)= 0$.

For offensive $k-$alliances, we define two types of alliance free (cover) sets depending on whether or not the boundary vertices of an offensive alliance affect the definition of the set. A set $S \subseteq V$ is *offensive $k-$alliance free* ($k-$oaf) if for all offensive $k-$alliances A, $A - S \neq \emptyset$. S is *weak offensive $k-$alliance free* ($k-$woaf) if for all offensive $k-$alliances A, $(A \cup \partial A) - S \neq \emptyset$. Similarly, a set $T \subseteq V$ is an *offensive $k-$alliance cover* ($k-$oac) if for all offensive $k-$alliances A, $A \cap T \neq \emptyset$. T is a *weak offensive $k-$alliance cover* ($k-$woac) if for all offensive $k-$alliances A, $(A \cup \partial A) \cap T \neq \emptyset$. The maximum (weak) offensive $k-$alliance free sets and minimum (weak) offensive $k-$alliance cover sets are defined in the same fashion as their defensive counterparts. For a graph G, we define the following invariants

- $\phi_k(G)$ = Size of a maximum $k-$daf set of G
- $\zeta_k(G)$ = Size of a minimum $k-$dac set of G
- $\phi_k^o(G)$ = Size of a maximum $k-$oaf set of G
- $\zeta_k^o(G)$ = Size of a minimum $k-$oac set of G
- $\phi_k^w(G)$ = Size of a maximum $k-$woaf set of G
- $\zeta_k^w(G)$ = Size of a minimum $k-$woac set of G

In the remaining part of this chapter, we explore the properties and bounds of the above defined invariants and their relationship with each other. In general we will refer to both offensive and defensive $k-$alliances as $k-$alliances. Similarly, the terms $k-$alliance free set and $k-$alliance cover set will encompass all types of alliance free sets and cover sets defined in this section.

4.2 Basic Properties

We begin by presenting some basic properties of the alliance free sets and cover sets.

Theorem 60 *$X \subseteq V$ is a $k-$alliance cover if and only if $V - X$ is $k-$alliance free.*

Proof. A set X is a defensive $k-$alliance free set if and only if, for every defensive $k-$alliance A, $A - X \neq \emptyset$ if and only if, for every defensive $k-$alliance A, $A \cap (V - X) \neq \emptyset$ if and only if $V - X$ is a defensive $k-$alliance cover.

The justification for (weak) offensive alliance covers is similar. □

Corollary 61 $\phi_k(G)+\zeta_k(G) = \phi_k^o(G) + \zeta_k^o(G) = \phi_k^w(G)+\zeta_k^w(G) = n(G)$

Corollary 62

(i) *If V' is a minimal k-dac ($k-$oac) then, $\forall v \in V'$, there exists a defensive (offensive) $k-$alliance S_v for which $S_v \cap V' = \{v\}$.*

(ii) *If V' is a minimal k-wdac then, $\forall v \in V'$, there exists an offensive $k-$alliance S_v for which $(S_v \cup \partial S_v) \cap V' = \{v\}$.*

Since, $\forall k_1 > k_2$, a k_2-alliance free set is also a k_1-alliance free set and every k_1-oaf set is also a k_1-woaf set, we have the following observation.

Observation 63 *For any graph G and $-\Delta < k_2 < k_1 \leq \Delta$,*

(i) $0 \leq \phi_{k_2}^o(G) \leq \phi_{k_1}^o(G) \leq \phi_{k_1}^w(G) \leq n(G)$

(ii) $0 \leq \phi_{k_1}^w(G) \leq \phi_{k_2}^w(G) \leq n(G)$

(iii) $0 \leq \phi_{k_2}(G) \leq \phi_{k_1}(G) \leq n(G)$

Also note that every $k-$daf set X is also a $k-$woaf set. Suppose not, then there is an offensive $k-$alliance A such that $A \cup \partial A \subseteq X$. Then $\forall v \in A' = A \cup \partial A$, $\deg_{A'}(v) \geq \deg_{V-A'}(v) + k$, which implies that A' is a defensive $k-$alliance and contradicts X being a $k-$daf set.

Observation 64 $\phi_k^w(G) \geq \phi_k(G)$

Suppose now a minimal k_1-dac set Y, $k_1 > -\delta(G)$, and let $A \subseteq Y$ such that A is an offensive k_2-alliance. Let $y \in A$. Then by Corollary 62, there exists a defensive k_1-alliance S_y such that $S_y \cap Y = \{y\}$. Hence $\exists x \in \partial A - Y$ such that $\deg_A(x) \leq \deg_{V-A}(x) + 2 - k_1$. Also, since A is an offensive k_2-alliance, $\deg_A(x) \geq \deg_{V-A}(x) + k_2$. Combining the two inequalities, we get, $k_2 \leq 2 - k_1$. This leads to the following observation:

Observation 65 *For any graph G and every k_1, k_2 such that $k_1 > -\delta(G)$ and $k_2 > 2 - k_1$,* $\phi_{k_2}^o(G) \geq \zeta_{k_1}(G)$

4.3 Defensive $k-$Alliance Free & Cover Sets

For any k, such that $-\delta(G) < k \leq \Delta(G)$, we know that any independent set in a connected graph G is $k-$daf, therefore $\phi_k(G) \geq \beta_0(G)$, where $\beta_0(G)$ is the vertex independence number of graph G. We can further improve this bound by noting that the addition of any $\left\lceil \frac{\delta(G)}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor - 1$ vertices to an independent set will not produce a defensive $k-$alliance in the new set, hence, $\phi_k(G) \geq \beta_0(G)+\left\lceil \frac{\delta(G)}{2} \right\rceil+\left\lfloor \frac{k}{2} \right\rfloor-1$. Since, every $A \subset V$, such that $|A| \geq n-\left\lfloor \frac{\delta(G)}{2} \right\rfloor+\left\lceil \frac{k}{2} \right\rceil$, is a defensive $k-$alliance, $\phi_k(G) < n - \left\lfloor \frac{\delta(G)}{2} \right\rfloor + \left\lceil \frac{k}{2} \right\rceil$.

Observation 66 *If G is a connected graph and $-\delta(G) < k \leq \Delta(G)$ then*

$$\beta_0(G) + \left\lceil \frac{\delta(G)}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor - 1 \leq \phi_k(G) < n - \left\lfloor \frac{\delta(G)}{2} \right\rfloor + \left\lceil \frac{k}{2} \right\rceil$$

Next we present the values of $\phi_k(G)$ for some common graph families.

Observation 67 *If G is an Eulerian graph and $-\frac{\delta(G)}{2} < i \leq \frac{\Delta(G)}{2}$, then $\phi_{2i-1}(G) = \phi_{2i}(G)$.*

Observation 68 *For the complete graph K_n and $-n+1 < k < n$,*

$$\phi_k(K_n) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{k}{2} \right\rceil & \text{for odd } n \\ \\ \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor & \text{for even } n. \end{cases}$$

Observation 69 *For the complete bipartite graph $K_{p,q}$, where $p \leq q$ and $-p < k \leq p$,*

$$\phi_k(K_{p,q}) = \begin{cases} q + \left\lceil \frac{p}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor - 1 & \text{for odd } p \\ \\ q + \left\lceil \frac{p}{2} \right\rceil + \left\lceil \frac{k}{2} \right\rceil - 1 & \text{for even } p. \end{cases}$$

Note that the upper and lower bounds of Observation 66 coincide for both K_n and $K_{p,q}$, when k is even. We now show that for $k \geq 0$, even complete graphs achieve the lower bound for $\phi_k(G)$.

To show this, we first present a bound on $\phi_k(G)$ when $k = 0$. The result is then generalized to $k \geq 0$ in Theorem 78.

Theorem 70 *If G is a connected graph then* $\phi_0(G) \geq \left\lfloor \frac{n}{2} \right\rfloor$.

Proof. Let A be a $\phi_0(G)-$set of a connected graph G and assume, to the contrary, that $\phi_0(G) < \left\lfloor \frac{n}{2} \right\rfloor$. Let $B = V(G) - A$, hence $|B| = \zeta_0(G) > \left\lceil \frac{n}{2} \right\rceil$. Since B is a $0-$dac, $\forall v \in B$ there exists a defensive $0-$alliance $S(v)$ such that $S(v) \cap B = \{v\}$. Hence, $\forall v \in B$, $\deg_A(v) \geq \deg_B(v)$. If B does not contain a defensive $0-$alliance, then B is a $0-$daf set, which is contradiction since, $|B| > \left\lceil \frac{n}{2} \right\rceil > \phi_0(G)$. Hence, B must contain a minimal defensive $0-$ alliance T. If $v \in T$ then $\deg_B(v) = \deg_A(v)$. Hence, $N_B(T) = T$.

Suppose T is the only minimal defensive $0-$alliance in B. Then, for any vertex $x \in T$, the set $B - \{x\}$ is a defensive $0-$alliance free set and $|B - \{x\}| > \phi_0(G)$, a contradiction. Thus there are at least two disjoint defensive $0-$alliances in B.

Now, assume that the number of disjoint minimal defensive $0-$alliances in B is minimum among all such sets. For each $v \in B$, let $S(v)$ be a minimal defensive $0-$alliance such that $S(v) \cap B = \{v\}$. Also, define:

$D = \{v \in B | \deg_B(v) = \deg_A(v)\}$,

$R = \{v \in A | \deg_A(v) = \deg_B(v)\}$,

$R^- = \{v \in A | \deg_A(v) < \deg_B(v)\}$, and

$R^+ = \{v \in A | \deg_A(v) > \deg_B(v)\}$.

Let $T_1, T_2, \ldots, T_r$ be the disjoint minimal defensive $0-$alliances in B. By the above arguments, $r \geq 2$ and $\forall i$, $N_B(T_i) = T_i \subseteq D$.

We now present a sequence of lemmas which culminate in the rest of the proof of Theorem 70.

Lemma 71 *For $1 \leq i \leq r$ and each $x \in T_i$, $N_A(x) \subseteq S(x) \cap R^-$.*

Proof. Suppose $x \in T_i$ and let $y \in N_A(x)$. Since $x \in T_i \subseteq D$, $\deg_B(x) = \deg_A(x)$. Hence, $N_A(x) \subseteq S(x)$ and $y \in S(x)$. Assume to the contrary that $y \notin R^-$, i.e., $\deg_A(y) \geq \deg_B(y)$. Let $A' = A \cup \{x\} - \{y\}$ and suppose $S' \subseteq A'$ is a defensive $0-$alliance. Since $\deg_{A'}(x) < \deg_{B'}(x)$, $x \notin S'$. But, then $S' \subseteq A$, which contradicts A being $0-$daf. Hence, A' is defensive $0-$alliance free and $B' = V - A'$ is a $0-$dac. Since T_i is a minimal defensive $0-$alliance in B, $T_i - \{x\}$ is not a defensive $0-$alliance in B'. Also, $\deg_{B'}(y) < \deg_{A'}(y)$ implies that $y \notin T'$, where T' is a defensive $0-$alliance in B'. But then the number of disjoint minimal defensive $0-$alliances in B' is $r-1$, which contradicts the assumption that B has a minimum number of disjoint minimal defensive $0-$alliances. $\square$

Lemma 72 *For $i \neq j$ and every $x_1 \in T_i$ and $x_2 \in T_j$, $N(x_1) \cap N(x_2) = \emptyset$.*

Proof. Suppose $i \neq j$ and there exist $x_1 \in T_i$ and $x_2 \in T_j$ such that $y \in N(x_1) \cap N(x_2)$. Since $T_i \cap T_j = \emptyset$ and $N_B(T_i) = T_i$, we have that $y \in A$. From Lemma 71, we know that $y \in R^- \cap S(x_1) \cap S(x_2)$. Consider the sets $A' = A \cup \{x_1, x_2\} - \{y\}$ and $B' = V - A'$. Since $|A'| = |A| + 1$ and A is a $\phi_0(G)-$set, A' must contain a defensive $0-$alliance S'. However, $\deg_A(x_l) = \deg_B(x_l)$, $l \in \{1, 2\}$ and $x_1x_2 \notin E(G)$. Therefore, $\deg_{A'}(x_l) = \deg_{B'}(x_l) - 1$ and, hence, $\{x_1, x_2\} \cap S' = \emptyset$. This implies that $S' \subseteq A$, and contradicts A being a defensive $0-$alliance free set. $\square$

Lemma 73 *For every $x \in T_i$*

(i) $S(x) \subseteq N_A(x) \cup R \cup \{x\}$,

(ii) $S(x)$ *is the unique minimal defensive* $0-$*alliance in* $A \cup \{x\}$, *and*

(iii) $N_{A\cup\{x\}}(S(x)) = S(x)$.

Proof. Let $x \in T_i$ and perform the following procedure:

$S' \leftarrow N_A(x) \cup \{x\}$

While $N_A(S') \subseteq N_A(x) \cup R$ *and* $N_A(S') - S' \neq \emptyset$

Begin

$S' \leftarrow S' \cup N_A(S')$

End

Since G is finite, the procedure will terminate with either $N_A(S') - S' = \emptyset$, or with a vertex $z \in N_A(S') - S'$ such that $z \notin R$. Assume $N_A(S') - S' \neq \emptyset$. By construction, $S' \cup N_A(S') \cup \{z\} \subseteq S(x)$ for every $S(x)$ that is a defensive $0-$alliance and for which $S(x) \cap B = \{x\}$. There are two cases.

Case 1. $z \in R^-$: This implies that $\deg_{A\cup\{x\}}(z) < \deg_{B-\{x\}}(z)$ and contradicts the assumption that $S(x)$ is a defensive $0-$alliance containing z.

Case 2. $z \in R^+$: The set $A' = (A \cup \{x\}) - \{z\}$ is a $\phi_0(G)-$set, otherwise there is a defensive $0-$alliance in $A \cup \{x\}$ not containing z. Thus, $B' = V - A'$ is a $0-$dac. Since T_i is a minimal defensive $0-$alliance in B, $T_i - \{x\}$ is not a defensive $0-$alliance in B'. Also, $\deg_{B'}(z) < \deg_{A'}(z)$ implies that $z \notin T'$, where T' is a defensive $0-$alliance in B'. But, then the number of disjoint minimal defensive $0-$alliances in B' is $r-1$, contradicting the assumption that B has minimum number of disjoint minimal defensive $0-$alliances.

Since both cases lead to a contradiction, we conclude that $N_A(S') - S' = \emptyset$. Hence, $S' = S(x) \subseteq N_A(x) \cup R \cup \{x\}$ and, by the construction, $S(x) = S'$ is the unique minimal defensive $0-$alliance in $A \cup \{x\}$. Also, since $v \in S(x)$ implies $\deg_{A\cup\{x\}}(v) = \deg_{B-\{x\}}(v)$, we must conclude that $N_{A\cup\{x\}}(S(x)) = S(x)$. □

Lemma 74 *For $i \neq j$ and every $x_1 \in T_i$ and $x_2 \in T_j$, $S(x_1) \cap S(x_2) = \emptyset$.*

Proof. Suppose $i \neq j$, $x_1 \in T_i$, and $x_2 \in T_j$. Assume, to the contrary, that $z \in S(x_1) \cap S(x_2)$. By Lemmas 71, 72 and 73, we know that $N_A(x_1) \subseteq S(x_1) \cap R^-$, $N_A(x_1) \cap N_A(x_2) = \emptyset$, and $S(x_2) \subseteq N_A(x_2) \cup R \cup \{x_2\}$. Hence, $N_A(x_1) \cap S(x_2) = \emptyset$. Since $S(x_1)$ is a minimal defensive $0-$alliance, $G[S(x_1)]$, the subgraph of G induced by $S(x_1)$, is connected. Hence, there is a path P in $G[S(x_1)]$ between z and a vertex $y \in N_A(x_1)$ that does not contain x_1. From Lemma 73, $N_{A\cup\{x_2\}}(S(x_2)) = S(x_2)$ and, hence, $y \in N_A(x_1) \cap S(x_2)$, a contradiction.

□

Corollary 75 *For $i \neq j$ and any $x_1 \in T_i$ and $x_2 \in T_j$, every path between $S(x_1)$ and $S(x_2)$ contains a vertex not in A.*

Lemma 76 *If $i \neq j$ then there is no path between T_i and T_j.*

Proof. Assume to the contrary that such a path exists. Recall that $T_i \cap T_j = \emptyset$ and $N_B(T_i) = T_i$. Hence, any path P from T_i to T_j must have an even number of edges in common with the edge cutset $\langle A, B\rangle$. Let the number of common edges between the edge cutset $F = \langle A, B\rangle$ and the path P be $|F \cap P| \geq 2$ and assume that $|F \cap P|$ is minimum for all such bipartitions. Now we have two cases:

Case 1: $|F \cap P| = 2$. Let $F \cap P = \{x_1a_1, a_2x_2\}$, where $x_1 \in T_i$, $a_1 \in N_A(x_1) \subseteq S(x_1)$, $x_2 \in T_j$, and $a_2 \in N_A(x_2) \subseteq S(x_2)$. By Lemma 74, $S(x_1) \cap S(x_2) = \emptyset$ and, by Corollary 75, there is no path from $S(x_1)$ to $S(x_2)$ consisting of only vertices in A, a contradiction.

Case 2: $|F \cap P| > 2$. Let $F \cap P = \{x_1a_1, a_2x_2, x_3a_3, \ldots, a_{2s+2}x_{2s+2}\}$, $s \geq 1$, where $x_1 \in T_i$, $a_1 \in N_A(x_1)$, $a_2 \in S(x_1)$, $x_2 \in N_B(a_2)$, $\ldots$, $a_{2s+2} \in N_A(x_{2s+2})$ and $x_{2s+2} \in T_j$. Further, for $1 \leq l \leq 2s+2$, $a_l \in A$ and $x_l \in B$. We claim for $2 \leq l \leq 2s+1$, that $x_l \notin T_u$, $1 \leq u \leq r$. Otherwise, suppose that $x_l \in T_u$. Without loss of generality, assume $u \neq i$, then there is a path from T_i to T_u such that $|F \cap P| \leq 2s$, which is contrary to P minimizing $|F \cap P|$.

Since $a_2 \in S(x_1)$, by Lemma 73, the set $A' = A \cup \{x_1\} - \{a_2\}$ is a $\phi_0(G)-$set and the set $B' = V - A'$ is a $0-$dac. Let $F' = \langle A', B' \rangle$. Suppose there is no defensive $0-$alliance T' in B' such that $a_2 \in T'$. Then there are $r-1$ disjoint minimal defensive $0-$alliances in B', which is a contradiction since B has the minimum number of disjoint minimal defensive $0-$alliances. Thus, there is a defensive $0-$alliance $T' \subseteq B'$ which contains a_2 and is disjoint from sets $T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_k$. But, then there is a path P' between T' and T_j such that $|F' \cap P'| = 2s$, which is again a contradiction.

Since both cases lead to contradictions, there is no path P between T_i and T_j whenever $i \neq j$. $\Box$

From Lemma 76, we conclude that G is disconnected, a contradiction. Therefore, the set B must be defensive $0-$alliance free and, hence, $\phi_0(G) \geq |B| > |A| = \phi_0(G)$, again a contradiction. Thus, $\phi_0(G) \geq \lfloor \frac{n}{2} \rfloor$, which completes the proof of Theorem 70. $\square$

Corollary 77 *If G is a connected Eulerian graph then $\phi_{-1}(G) \geq \lfloor \frac{n}{2} \rfloor$.*

In Chapter 5, we show that for connected graphs G, $\phi_0(G) < \zeta_0(G)$ if and only if every block of G is either an odd clique or an odd cycle.

Theorem 78 *For every connected graph G and $0 \leq k \leq \Delta$, $\phi_k(G) \geq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{k}{2} \rfloor$.*

Proof. By Theorem 70, the statement is true for $k = 0$. Since every $k-$daf set is also $(k+1)-$daf, $\phi_1(G) \geq \phi_0(G) \geq \lfloor \frac{n}{2} \rfloor = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{1}{2} \rfloor$, i.e., the statement is also true for $k = 1$. Hence, we may proceed by induction on k.

Assume that the statement is true for $k \leq M$ for arbitrary $M > 1$. Let A be a $\phi_M(G)-$set of a graph G. Again, A is also $(M+2)-$daf of graph G. By the induction hypothesis, $\phi_{M+2}(G) \geq |A| = \phi_M(G) \geq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{M}{2} \rfloor$. If there exists a vertex $v \in V - A$ such that $A \cup \{v\}$ is $(M+2)-$daf, then $\phi_{M+2}(G) \geq |A \cup \{v\}| \geq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{M}{2} \rfloor + 1 = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{M+2}{2} \rfloor$. Suppose no such vertex exists. Then, $\forall v \in V - A$ there exists a defensive $(M+2)-$alliance $S(v)$ such that $S(v) \cap (V - A) = \{v\}$. But, then $\forall w \in S(v)$, $\deg_{S(v)-\{v\}}(w) \geq \deg_{V-S(v)-\{v\}}(w) + M$ which is contrary to the assumption that A is $M-$daf. $\square$

The bound of Theorem 78 is also sharp and is achieved by the complete graphs of even order. We believe (but have been unable to prove) the following extension of the above theorem:

Conjecture 79 *If G is a connected graph and $-\delta(G) < k \leq \delta(G)$ then*

$$\phi_k(G) \geq \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor.$$

Next, we show that no forbidden subgraph characterization exists for the graphs induced by minimal $k-$dac sets.

Theorem 80 *Let G be any graph and r an integer such that $r \geq 2$. Then, for all $k \geq 2-r$, there is a graph G', such that G' contains G as an induced subgraph and $\zeta_k(G') = r$.*

Proof. Let a graph $G = (V, E)$ where $V = \{v_1, v_2, \ldots, v_n\}$ and construct a graph $G' = (V', E')$ as follows: $V' = V \cup X \cup Y$, where $X = \left\{x_i^j,\ 1 \leq i \leq n,\ 1 \leq j \leq \max(2r+k, \Delta(G) - k + 1)\right\}$ and $Y = \{y_1, y_2, \ldots, y_{2r+k-2}\}$. $E' = E \cup E_1 \cup E_2$, where $E_1 = \left\{v_i x_i^j,\ v_i \in V,\ x_i^j \in X\right\}$ and $E_2 = \left\{x_i^j y_l,\ x_i^j \in X,\ y_l \in Y\right\}$. Thus $\delta(G') = 2r + k - 1$. Since by Observation 66, $\zeta_k(G') \geq \left\lfloor \frac{\delta(G')}{2} \right\rfloor - \left\lceil \frac{k}{2} \right\rceil + 1$, we have $\zeta_k(G') \geq \left\lfloor \frac{2r+k-1}{2} \right\rfloor - \left\lceil \frac{k}{2} \right\rceil + 1 = r$.

Now consider $C \subseteq Y$ such that $|C| = r$. We claim that C is a $k-$dac set of graph G'. Suppose not. Then there exists a defensive $k-$alliance $S \subseteq V' - C$ in G'. Let $v \in S$. Since $\forall x \in X,\ \deg(x) = 2r + k - 1$, if $v \in X$ then $\deg_S(v) \leq r + k - 1 < \deg_C(v) + k = r + k$, which is contrary to S being a defensive $k-$alliance. Hence $S \cap X = \emptyset$. Now let $v \in V$. By construction of graph G', $\forall v \in V,\ \deg_X(v) + k \geq \Delta(G) + 1 > \deg_{V'-X}(v) \geq \deg_S(v)$, again a contradiction. The only remaining case is $S \subset Y$, which is not possible as $\forall v \in S,\ \deg_S(v) = 0 < deg_{V'-S}(y) + k \leq n(2r+k) + k$. Hence $S = \emptyset$ and C is a $k-$dac set. Thus $\zeta_k(G') \leq r$.

Combining the two results, we get $\zeta_k(G') = r$. $\square$

4.4 Offensive $k-$Alliance Free & Cover Sets

In this section, we study the properties of the free sets and cover sets associated with offensive $k-$alliances. We begin by presenting the values of $\phi_k^o(G)$ and $\phi_k^w(G)$ for some special classes of graphs.

Observation 81 *For the complete graph K_n, and $-n+3 < k < n$*

$$\phi_k^o(K_n) = \phi_k(K_n) - 1 = \left\lfloor \frac{n+k}{2} \right\rfloor - 1$$

$$\phi_k^w(K_n) = n - 1$$

Observation 82 *For the complete bipartite graph $K_{p,q}$, $p \leq q$, and $-p+2 < k \leq q$*

$$\phi_k^o(K_{p,q}) = \begin{cases} \left\lceil \frac{q}{2} \right\rceil + \left\lceil \frac{p}{2} \right\rceil + 2\left\lfloor \frac{k}{2} \right\rfloor - 2 & p \text{ and } q \text{ both odd} \\ \left\lceil \frac{q}{2} \right\rceil + \left\lceil \frac{p}{2} \right\rceil + 2\left\lceil \frac{k}{2} \right\rceil - 2 & p \text{ and } q \text{ both even} \\ \left\lceil \frac{q}{2} \right\rceil + \left\lceil \frac{p}{2} \right\rceil + k - 2 & \text{otherwise} \end{cases}$$

$$\phi_k^w(K_{p,q}) = n - 2, \; p, q \neq 1$$

It is interesting to note that while complete graphs attain the lower bound for $\phi_k(G)$, they have the maximum value for $\phi_k^w(G)$.

Lemma 83 *If S is an offensive k_1- alliance then*

(i) *for all offensive* k_2-*alliances* $S' \subseteq V - S$ *such that* $k_1 + k_2 > 0$, $\partial S \cap \partial S' = \emptyset$.

(ii) *for all defensive* k_2-*alliances* $S' \subseteq V - S$ *such that* $k_1 + k_2 > 0$, $\partial S \cap S' = \emptyset$.

Theorem 84 *For a connected graph* G*, if* X *is a maximal* k_1-*woaf set and* $Y = V - X$ *then*

(i) $\forall k_2 > -k_1$, Y *is a* k_2-*woaf set (and hence,* X *is a* k_2-*woac set), and*

(ii) $\forall k_2 > \max(-k_1, -\delta(G))$, Y *is a* k_2-*daf set (hence,* X *is a* k_2-*dac set).*

Proof. For i), let $k_2 > -k_1$ and suppose there exists an offensive k_2-alliance S for which $N[S] \subseteq Y$. Let $x \in \partial S$. From Corollary 62, there is an offensive k_1-alliance S_x for which $N[S_x] \cap Y = \{x\}$. If $x \in \partial S_x$, then from Lemma 83, S and S_x cannot be disjoint, a contradiction. So we must assume that $x \in S_x$. But then, $N(x) \subseteq \partial S_x \subseteq X$, which leads to a contradiction since x must have at least one neighbor in $S \subseteq Y$. Thus, Y is a k_2-woaf set and, from Theorem 60, X is a k_2-woac set.

For ii), let $k_2 > \max(-k_1, -\delta(G))$ and suppose there exists a defensive k_2- alliance $S \subseteq Y$. Let $x \in S$. From Corollary 62, there exists an offensive k_1- alliance S_x for which $N[S_x] \cap Y = \{x\}$. If $x \in \partial S_x$ then from Lemma 83, S and S_x cannot be disjoint, a contradiction. So we must assume that $x \in S_x$, but then $N(x) \subseteq \partial S_x \subseteq X$, which is not possible since $\deg_S(x) \geq (deg(x) + k_2)/2 > 0$. Hence, Y is a k_2-daf set and, from Theorem 60, X is a k_2-dac set. $\square$

Corollary 85

(i) *Every maximal k_1-woaf set contains a minimal k_2-woac set,* $\forall k_2 > -k_1$.

(ii) *Every maximal k_1-woaf set contains a minimal k_2-dac set,* $\forall k_2 > \max(-k_1, -\delta(G))$.

Since every $k-$woaf is also $l-$woaf $\forall l > k$, by Theorem 60, every $k-$woac is also $l-$woac. This observation leads to the following corollary of Theorem 84.

Corollary 86 $\forall k > 0,\ \zeta_k^w(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$

It is easy to prove that $\forall k \geq 0$, $\zeta_k^w(G) = \left\lfloor \frac{n}{2} \right\rfloor$ if and only if $G \approx K_2$ and $k < 2$.

We conclude this section by presenting a result for $\zeta_k^w(G)$ similar to the one for $\zeta_k(G)$ in Theorem 80.

Theorem 87 *Let G be any graph and r an integer such that $r \geq 1$. Then there is a graph G' with $\zeta_k^w(G') = r$, which contains G as an induced subgraph.*

Proof. Let a graph $G = (V, E)$ where $V = \{v_1, v_2, \ldots, v_n\}$ and construct a graph $G' = (V', E')$ as follows: $V' = V \cup X \cup Y$, where $X = \{x_1, x_2, \ldots, x_r\}$ and Y is the union of disjoint sets $Y_1, Y_2, \ldots, Y_r$, such that $\forall i,\ |Y_i| = n - k + 1$. $E' = E \cup E_1 \cup E_2 \cup E_3$, where $E_1 = \{v_i x_j,\ v_i \in V,\ x_j \in X\}$, $E_2 = \bigcup_{i=1}^{r} \{x_i y,\ \forall y \in Y_i\}$ and $E_3 = \{yz|\ y, z \in Y_i,\ 1 \leq i \leq r\}$. Hence, G' is obtained by adding r vertex disjoint cliques $Y_i \cup \{x_i\}$, each of order $n - k + 2$ vertices and making each x_i adjacent to every vertex of V.

It is easy to see that X is a $k-$woac set of graph G', i.e. $\zeta_k^w(G') \leq |X| = r$. We claim that $\zeta_k^w(G') = r$. Suppose not and let $C \subset V$ be a $k-$woac set of graph G' such that $|C| < r$.

By pigeon hole principle, there exists Y_i such that $(Y_i \cup \{x_i\}) \cap C = \emptyset$. Since $\partial Y_i = \{x_i\}$ and $deg_{Y_i}(x_i) = n+k+1 > deg_{V'-Y_i}(x_i)+k = n+k$, Y_i is an offensive $k-$alliance in G' such that $N[Y_i] \subseteq V' - C$, which is contrary to C being a $k-$woac set of graph G'. Hence $\zeta_k^w(G') \geq r$.

Combining the two results, we get $\zeta_k^w(G') = r$. $\square$

CHAPTER 5

PARTITIONING A GRAPH INTO DEFENSIVE 0-ALLIANCE FREE (COVER) SETS

In this chapter, we deal with the problem of partitioning the vertex set of a graph G into defensive 0-alliance free sets. We refer to such a partition as a *defensive 0-alliance-free partition* and say G is partitionable if it has a defensive 0-alliance-free partition. Recall from Chapter 3 that a partition is said to be unfriendly if each vertex has as many or more neighbors outside the set in which it occurs than inside it. Note that, in an unfriendly partition, if every vertex has *strictly* more neighbors outside the set in which it occurs than inside it, then the partition is a defensive 0-alliance-free partition. However, the reverse is not true, i.e., a vertex in a defensive 0-alliance free partition may have the same number of neighbors inside the set in which it occurs than outside it.

As in the case for satisfactory partitions, not all the graphs have a defensive 0-alliance-free partition. For example, odd cliques and odd cycles do not have defensive 0-alliance-free partitions. In this chapter, we characterize graphs having defensive 0-alliance-free partitions. In particular, we show the following:

Theorem 88 *A connected graph G is partitionable if and only if G has a block that is other than an odd clique or an odd cycle.*

Define a set S to be a *defensive 0-alliance free cover* if S is both defensive 0-alliance free and a defensive 0-alliance cover. Equivalently, S is a defensive 0-alliance free cover if for all alliances X, $X \cap S \neq \emptyset$ and $X \cap (V - S) \neq \emptyset$. Thus, we have the following:

Lemma 89 *A set S is a defensive 0-alliance free cover if and only if $V - S$ is a defensive 0-alliance free cover.*

From Lemma 89 and Theorem 60, we conclude the following:

Theorem 90 *A graph G is partitionable if and only if G has a defensive 0-alliance free cover.*

5.1 When G is not Partitionable

We call a defensive 0-alliance cover X to be *special* if X contains exactly one minimal defensive 0-alliance U_X, such that:

1. $\forall x \in U_X$, $\deg_X(x) = \deg_{V-X}(x)$, i.e., $N_X(U_X) = U_X$, and
2. $\forall x \in U_X$, $(V - X) \cup \{x\}$ is also a special defensive 0-alliance cover.

It is shown in [SD02b] that if G does not have an defensive 0-alliance free cover then it has a special defensive 0-alliance cover. Hence, from Theorem 90, if a graph is not partitionable, it must contain a special defensive 0-alliance cover. The following lemma is immediate from the definition of special defensive 0-alliance cover.

Lemma 91 *If G is not partitionable and X is a special defensive 0-alliance cover in G then for any $x \in U_X \subseteq X$ and $y \in U_{(V-X)\cup\{x\}}$, $X' = (X - \{x\}) \cup \{y\}$ is a special defensive 0-alliance cover, and $y \in U_{X'}$.*

Lemma 92 *If G is not partitionable then for every $v \in V(G)$, there exists a special defensive 0-alliance cover X such that the minimal defensive 0-alliance U_X contains v.*

Proof. Assume to the contrary, and let $x \in V(G)$, such that for every special defensive 0-alliance cover X, $x \notin U_X$. Let $v \in U_X$ be a nearest vertex to x. Also, let $P = v, v_1, v_2, \ldots, v_k, x$ be a shortest path from x to v. Since $N_X(U_X) = U_X$, and P has minimum length, $v_1 \in V - X$. By the definition of special defensive 0-alliance cover, $Y = (V - X) \cup \{v\}$ is a special defensive 0-alliance cover, and $v_1 \in U_Y$, which is contrary to v being a nearest such vertex to x. □

Corollary 93 *If G is not partitionable, then G is Eulerian.*

Proof. By definition of special defensive 0-alliance cover, if U_X is the minimal defensive 0-alliance in an special defensive 0-alliance cover X, then $\forall x \in U_X$, $\deg_X(x) = \deg_{V-X}(x)$. From Lemma 92, every vertex $v \in V$ is in some such U_X, hence, every vertex in G must have even degree, i.e., Eulerian. □

The following theorem describes the partitionable graphs in terms of their blocks.

Theorem 94 *A connected graph G is partitionable if and only if some block of G is partitionable.*

Proof. The proof is by induction on the number of blocks in graph G. The statement is true if G is itself a block, and hence, the base case is true. Assume that the statement is true for all graphs with at most r blocks, for a fixed but arbitrary $r \geq 1$. Consider a graph G with $r+1$ blocks and let x be a cut-vertex in G. Let G_1 be the graph induced by $V_1 \subset V$, where $x \in V_1$ and $V_1 - \{x\}$ induces a connected component in graph $G - \{x\}$. Also, let G_2 be the graph induced by $V_2 = (V - V_1) \cup \{x\}$.

First, assume that G is partitionable and thus has a defensive 0-alliance free cover, say B'. Further, assume that neither G_1 nor G_2 is partitionable. From Lemma 89, we may assume that $x \in B'$. Note that for $i \in \{1, 2\}$, $B_i = B' \cap V_i$ is defensive 0-alliance cover in graph G_i. Thus each B_i must contain a defensive 0-alliance T_i in graph G_i. Now we have two cases. Case 1: For some $i \in \{1, 2\}$, $x \notin T_i$. Then, $T_i \subseteq B'$ is also a defensive 0-alliance in graph G, which is contrary to B' being a defensive 0-alliance free cover in graph G. Case 2: $x \in T_1 \cap T_2$. But then, $T_1 \cup T_2 \subseteq B'$ is a defensive 0-alliance in graph G, again a contradiction.

Since both cases lead to a contradiction, we conclude that at least one of G_1 and G_2 is partitionable. Thus, by induction hypothesis, some block of G_1 or G_2 is partitionable. Hence, some block of G is partitionable.

Next, suppose some block of G is partitionable. We may assume without loss of generality that the block is in G_1 and, hence, by the induction hypothesis, G_1 is partitionable. Let B_1

be a defensive 0-alliance free cover in G_1. From Lemma 89, we may assume that $x \notin B_1$. There are two cases. Case 1: G_2 is partitionable. Then, there is a defensive 0-alliance free cover B_2 in G_2. Once again, we may assume that $x \notin B_2$. But then $B_1 \cup B_2$ is a defensive 0-alliance free cover of graph G, thus G is partitionable. Case 2: If G_2 is not partitionable, every defensive 0-alliance cover in G_2 contains some defensive 0-alliance. By Lemma 92, there exists a special defensive 0-alliance cover B_2 in G_2, such that $x \in U_{B_2}$. If $B' = (B_1 \cup B_2) - \{x\}$ is not a defensive 0-alliance cover of graph G then there must exist a defensive 0-alliance S in G, such that $S \cap B' = \emptyset$ and $x \in S$. Since $x \in U_{B_2}$, $|N_{V_2 \cap S}(x)| = |N_{V_2 - S}(x)|$. From Corollary 93, we may assume that G is Eulerian, and $|N_{V_1}(x)| \geq 2$, hence, $V_1 \cap S \neq \emptyset$ and $|N_{V_1 \cap S}(x)| \geq |N_{V_1 - S}(x)|$. But then, $V_1 \cap S$ is also a defensive 0-alliance in graph G_1, which contradicts B_1 being a defensive 0-alliance cover in G_1. Hence, B' is a defensive 0-alliance free cover of graph G, and G is partitionable. □

5.2 When a Block is Not Partitionable

From Theorem 94, a graph is not partitionable if and only if every block of G is not partitionable. In this section, we characterize the blocks that are not partitionable.

Let G be a block that is not partitionable, and let X be a special defensive 0-alliance cover in G containing a defensive 0-alliance U_X. Also let $Y = V - X$.

Lemma 95 *If a block G is not partitionable block then the graph $G[U_X]$ is a block.*

Proof. Assume to the contrary that x is a cut vertex in $G[U_X]$. Let $\{a, b\} \subseteq U_X$, such that every $a-b$ path in $G[U_X]$ contains x. Since G is a block, there must be a path P in G from a to b that does not contain x. Since $N_X(U_X) = U_X$, $P \cap \langle X, Y\rangle \neq \emptyset$. Assume now that the choice of X, x, a and b is such that $|P \cap \langle X, Y\rangle|$ is minimum among all such choices. Further, assume that P is a shortest such path in G. Let $P \cap \langle X, Y\rangle = \{y_1y_2, y_3y_4, \ldots, y_{4k-1}y_{4k}\}$ for some $k \geq 1$, where $\{y_{4i-3}, y_{4i}\} \subseteq X$ and $\{y_{4i-2}, y_{4i-1}\} \subseteq Y$, $1 \leq i \leq k$. In addition, y_{2j} may be the same as y_{2j+1}, $0 < j < 2k$. Since P is a shortest such path, $y_1 = a$ and $y_{4k} = b$. Let $X_0 = X$ and for $1 \leq i \leq k$, define;

$X_i = (X_{i-1} - \{y_{4i-3}\}) \cup \{y_{4i-1}\}$, and

$Y_i = V - X_i$.

From Lemma 91, $\forall i$, X_i is a special defensive 0-alliance cover. Also, $\forall i > 0$, $\{y_{4i-1}, y_{4i}, y_{4i+1}\} \subseteq U_{X_i}$ and $y_{4i-1}y_{4i} \in E(G)$.

Let $U' \subseteq U_{X_0}$, such that $G[U']$ is a connected component in $G[U_{X_0} - a]$ and $b \in U'$. Note that, $\forall v \in U' - N(a)$, $\deg_{U'}(v) = \deg_{V-U'}(v)$. In particular, $\deg_{U'}(b) = \deg_{V-U'}(b)$. Now there are two cases:

Case 1: Either $k = 1$ or for all j, $0 < j < k$, $U' \cap U_{X_j} = \emptyset$. Since $b \in U_{X_k}$ and $N(U_{X_k}) = U_{X_k}$, $U' \subseteq U_{X_k}$. But $\deg_{U'}(b) = \deg_{V-U'}(b)$, and $y_{4k-1}b \in E(G)$ imply that $\deg_{X_k}(b) > \deg_{Y_k}(b)$, which is contrary to X_k being a special defensive 0-alliance cover.

Case 2: For some j, $0 < j < k$, $U' \cap U_{X_j} \neq \emptyset$. Let j be the smallest such index. Since, $N_{X_j}(U_{X_j}) = U_{X_j}$, $U' \subseteq U_{X_j}$. Since $j < k$ and $|P \cap \langle X, Y\rangle|$ is minimum, every path in $G[U_{X_j}]$ from y_{4j-1} to b must contain x. But then, x is a cut vertex in $G[U_{X_j}]$ and

$|P' \cap \langle X_j, Y_j \rangle| < |P \cap \langle X, Y \rangle|$, where $P' \subseteq P$ is a path from y_{4j-1} to b that does not contain x, a contradiction.

Since both cases lead to contradiction, we conclude that, $G[U_X]$ is a block. □

Lemma 96 *If G is not partitionable and $\{u, v\} \subseteq U_X$, such that $N_{V-X}(u) \cap N_{V-X}(v) \neq \emptyset$ then $uv \in E(G)$.*

Proof. Let $\{u, v\} \subseteq U_X$, such that $z \in N_{V-X}(u) \cap N_{V-X}(v)$. By Lemma 91, $X' = (X - \{u\}) \cup \{z\}$ is a special defensive 0-alliance cover, and $z \in U_{X'}$. Since $v \in N_{X'}(z)$, $v \in U_{X'}$, i.e., $|N_{V-X'}(v)| = |N_{X'}(v)|$, which is possible only if $uv \in E(G)$. □

Lemma 97 *If a block G is not partitionable and X is a special defensive 0-alliance cover with $|U_X| > 2$ then for any $\{a, b\} \subset U_X$, $N_Y(a) \cap N_Y(b) \neq \emptyset$, where $Y = V - X$.*

Proof. Let $|U_X| > 2$ and $\{a, b\} \subseteq U_X$. From Lemma 95, $\forall x \in U_X$, $|N_{U_X}(x)| \geq 2$. Let $y_2 \in N_Y(a)$. Since G is a block, there must exist a path P from y_2 to b that does not contain a. Let P be such a path, for which $|P \cap \langle X, Y \rangle|$ is minimum among all such paths. Let $y_1 = a$ and $P \cap \langle X, Y \rangle = \{y_3y_4, y_5y_6, \ldots, y_{4k-1}y_{4k}\}$, $k \geq 1$, where $\{y_{4i-3}, y_{4i}\} \subseteq X$ and $\{y_{4i-2}, y_{4i-1}\} \subseteq Y$, $1 \leq i \leq k$. Further, y_{2j} may be the same as y_{2j+1}, $0 < j < 2k$. Also, let $y_{4k+1} = b$, $X_0 = X$ and for $1 \leq i \leq k$, define;

$X_i = (X_{i-1} - \{y_{4i-3}\}) \cup \{y_{4i-1}\}$, and

$Y_i = V - X_i$.

From Lemma 91, $\forall i$, X_i is a special defensive 0-alliance cover. Also, $\forall i > 0$, $\{y_{4i-1}, y_{4i}, y_{4i+1}\} \subseteq U_{X_i}$ and $y_{4i-1}y_{4i} \in E(G)$. Note that, $\forall i, 0 < i < k$, $U' \cap U_{X_i} = \emptyset$, where $U' = U_X - \{y_1\}$, otherwise, there is a $y_2 - b$ path $P' \subseteq P$ such that $|P' \cap \langle X, Y\rangle| < |P \cap \langle X, Y\rangle|$, a contradiction. Since $b \in U_{X_{2k}}$, $U' \subseteq U_{X_{2k}}$. Hence, $\forall z_i \in N_{U_X}(a)$, $y_{4k-1}z_i \in E(G)$. Since $|N_{U_X}(a)| > 1$, there are at least two vertices z_1, z_2 in U_X such that $y_{4k-1} \in N_Y(z_1) \cap N_Y(z_2)$. From Lemma 96, $z_1z_2 \in E(G)$.

We now claim that $\forall x \in U_X$, $y_{4k-1} \in N(x)$. Suppose not. Then there must exist $\{u, v, w\} \subseteq U_X$, such that $\{v, w\} \subseteq N(u)$, and $y_{4k-1} \in (N(u) \cap N(v)) - N(w)$. By Lemma 91, $X' = (X - \{u\}) \cup \{y_{4k-1}\}$ is a special defensive 0-alliance cover, and $y_{4k-1} \in U_{X'}$. Also, since $G[U_X]$ is a block and $N_{X'}(U_{X'}) = U_{X'}$, $N_{X'}(y_{4k-1}) = N_X(u)$, a contradiction. Hence, $\forall x \in U_X$, $y_{4k-1} \in N(x)$, which completes the proof. □

Theorem 98 *If G is a block, then G is partitionable if and only if G is neither an odd clique nor an odd cycle.*

Proof. It is easy to see that odd complete graphs and odd cycles are not partitionable. To prove the sufficiency of the theorem, let G be a block that is not partitionable and consider two exhaustive cases:

Case 1: There exists an special defensive 0-alliance cover X in G, such that $|U_X| > 2$. Let $Y = V - X$. From Lemmas 96 and 97, $G[U_X]$ is a clique, and $\forall x \in U_X$, $G[U_{Y\cup\{x\}}]$ is also a clique. Hence $\forall x \in U_X$, $N[x] = U_X \cup U_{Y\cup\{x\}}$. Also, from the definition of special defensive 0-alliance covers, $N_{Y\cup\{x\}}\left(U_{Y\cup\{x\}}\right) = U_{Y\cup\{x\}}$. Thus, from Lemma 97, for every $\{x, y\} \subset U_X$, $N[x] = N[y]$. By above arguments, $\forall x \in U_X$, $N[x]$ is a clique, and is a connected component of the graph G. Since G is connected, this is only possible if $G = G[N[x]]$. Hence, G is a complete graph. In addition, since even cliques are partitionable, G has odd order.

Case 2: For all special defensive 0-alliance covers X in G, $|U_X| = 2$. From Lemma 91, for all $w \in V$, there exists a special defensive 0-alliance cover B, such that $w \in U_B$. Since, $|U_B| = 2$ and $\deg_{U_B}(w) = \deg_{V-U_B}(w)$, $\deg(w) = 2$, and hence, G is a cycle. Further, since even cycles are partitionable, G is an odd cycle. □

From Theorems 94 and 98, we conclude that a connected graph G is partitionable if and only if G has a block that is other than an odd clique or an odd cycle, which is our main result (Theorem 88).

For an Eulerian graph, a defensive 0-alliance free set is also a defensive (-1)-alliance free set. Thus we have:

Corollary 99 *An Eulerian graph G has a partition into defensive (-1)-alliance free sets if and only if G has a block that is other than an odd clique or an odd cycle.*

The characterization of graphs with defensive (-1)-alliance free partition is still an open problem.

Also note that for $k > 0$, every unfriendly partition is a partition into defensive k-alliance free sets. Therefore we have the following result.

Theorem 100 *For $k > 0$, every graph G has a partition into defensive k-alliance free sets and this partition can be found in polynomial time.*

CHAPTER 6

GRAPH PARTITIONING AND DATA CLUSTERING

6.1 Introduction

Clustering is a process of partitioning a set of data into *clusters*, where a cluster is a collection of data points that are *similar* to each other and *dissimilar* to other data points. The problem and its many variants have been studied extensively in mathematics as well as in applied sciences. In the recent years, the availability of vast amounts of data (due to the emergence of the world wide web, enormous increase in computing power, data storage and communication speed) and the concept of *data mining* these massive databases has revitalized research on the problem. Other than that, the clustering concepts are widely applied in the areas of pattern recognition, machine learning and computer vision.

Different clustering algorithms use different concepts of cluster specific to the application for which the algorithm is used. Moreover, sometimes the term cluster is implicitly defined by the clustering criterion itself. In general, we define a set of clusters such that the similarities of objects within each cluster as well as the dissimilarities of objects among the clusters are maximized. Assume that the vertices in a graph are objects that we seek to group and the edges (or weight of the edges) define the common property (similarity) that the objects share. Clustering of these objects is then defined as a division of vertices into groups within which the edges are dense, but between which they are sparser. This in turn implies that the vertices in the groups have at least as many edges adjacent to the vertices inside the group as to the vertices outside it. Recall from chapter 2 that a strong defensive alliance is a set of vertices that have at least as many neighbors inside the set as they have outside it. Thus, each cluster is a strong defensive alliance.

Once the definition of cluster is agreed upon, one can pose several interesting questions, both of practical or theoretical natures. For example,

Q.1. How are the similarity measures to be defined? Similarity measures play an important part in determining the quality of clustering. Different techniques demand different types of measures, such as, measure of similarity between two objects, measure of similarity between an object and a set of objects, and measure of similarity between two sets of objects. Though similarity measures between objects are usually available for the problem in hand, similarity measures between two sets, *inter-cluster distances*, are hard to define. These distances are required by the clustering algorithms, which proceed by merging smaller clusters with small inter-cluster distances (*Agglomerative methods*) and/or splitting large insufficiently similar clusters into smaller clusters (*Divisive methods*). Standard inter-cluster distances include the distance between the closest elements of the two clusters (*single-link clustering*), the distance between the farthest elements of the two clusters (*complete-link clustering*), and the average distance between the elements in the clusters (*group average clustering*). For more examples of inter-cluster distances, see [Mir96]

Q.2. What is the best choice for the numbers of clusters [Dub87, Bou99]? This fundamental question in clustering is substantially hard to answer and is usually dodged by either i) clustering the data for different choice of number of clusters and use the cluster validity measures to decide which set of clusters best model the data, or ii) by generating clusters as nested structures called *Hierarchies*. Given a data set S, a hierarchy is a set H of subsets $S_w \subset S$, $w \in W$ (where W is an index set), called *clusters* and satisfying the following conditions: 1) $S \in H$ and 2) for any $S_1, S_2 \in H$, $S_1 \cap S_2 \in \{\emptyset, S_1, S_2\}$. Hence, a hierarchy of clusters consists of several levels, where the highest level contains a single cluster (the whole data), and each data item is considered as a separate cluster at the lowest level. A cluster in each intermediate level is partitioned by several clusters at lower levels. The final choice of clustering is then decided by analyzing this hierarchy.

Both of these strategies lead to our next four questions.

Q.3. What is the measure of goodness of a given clustering [HA85, Dav79, BP98]?

Q.4. Can the given data be partitioned into k clusters (for some given k)? Of particular interest is the special case, when $k = 2$ which provides the basis to most *divisive* clustering algorithms. This problem is also known as *Assessment of Clustering Tendency*, i.e., determining whether the data have structure in them or not without explicitly looking for clusters in the data [JD88, Eve93b].

Q.5. What is the maximum k for which such a partitioning exist?

Q.6. Given that there is a partition of k clusters in a given data, what is the best such clustering?

Q.7. Given k data items, is there a partition of k clusters, each containing one of the given items?

Q.8. What are the upper and lower bounds on the size of a cluster? What are the characteristics of extremal cases?

Q.9. Given a data item x, how many clusters contain x?

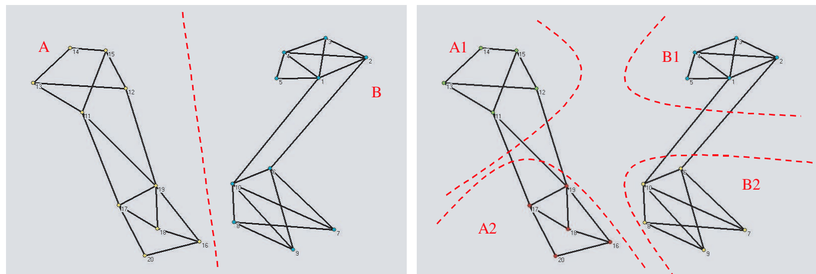
Q.10. How many clusters are there in a given data?

Q.11. What is the minimum size of a set containing at least one data item from each cluster?

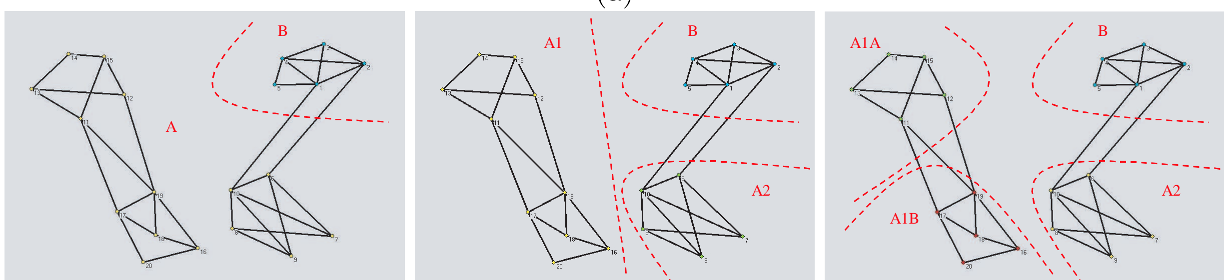
Q.12. What is the maximum size of a set that does not contain any cluster?

In this chapter, we use strong defensive alliances as a model of clusters in the data. With strong defensive alliance as the notion of cluster, the results presented in the previous chapters are directly related to some of the questions posed above. The problem of assessment of cluster tendency (Q.4), i.e., the existence of a bipartition into strong defensive alliances is studied in Chapter 3. The maximum number of clusters (Questions 4, 5 and 12) is bounded by the size of a minimum alliance cover of the graph (Chapter 4). The bounds on the size of a cluster (Q. 8) are presented in Chapter 2. In this chapter, we present an algorithmic solution for finding the clusters in a given data. Since the number of clusters are not known in advance (Q. 2), the proposed algorithm generates a hierarchy of clusters by splitting each cluster (starting with the cluster consisting of whole data) into two smaller clusters (strong defensive alliances) until no cluster can be further partitioned.

Note that a given set of data may have many partitions into strong defensive alliances and not every such partition is a good choice for generating a hierarchy of clusters. For example, in the case of a disconnected graph with two connected components, the better choice for the first bipartition is the one that divides the graph into two connected components, and not the one that separates a strong defensive alliance from any of the components. This problem is illustrated in Figure 6.1. In Figure 6.1(a), we first divide one of the connected components into two strong defensive alliances and then proceed by further dividing the generated clusters. In Figure 6.1(b), we first partition the graph into two components (strong defensive alliances), and then proceed by further dividing the components. Even though the final set of clusters is the same for both the cases, the hierarchies are not. In general, we want the algorithm to generate the best division of data at every level of the hierarchy. It is easy to see that at the first level of hierarchy, the partition of Figure 6.1(b) is a better choice than the partition of Figure 6.1(a). The hierarchies of Figure 6.1 are represented in the form of a tree or *dendrogram* in Figure 6.2.



(a)



(b)

Figure 6.1: (a) Two levels of a clustering hierarchy. In the first level the graph is split into two clusters A and B. In the second level, each of these clusters are further subdivided into two clusters. (b) Three levels of a clustering hierarchy. In the first level the graph is split into two clusters A and B. In the second level, cluster A is again split into two clusters A1 and A2. Cluster A1 is split into two more clusters in level 3.

From the above discussion, we conclude that, for each division, the algorithm must chose the best partition among all satisfactory partitions of the graph. To clearly define this problem, one has to define some measure of goodness for each partition (Q.3 above). Recall from Chapter 2, that for any p, $0 \leq p \leq 1$, a defensive p-alliance, S, is a set of vertices for which the ratio of the number of neighbors inside the set S and the size of neighborhood is at

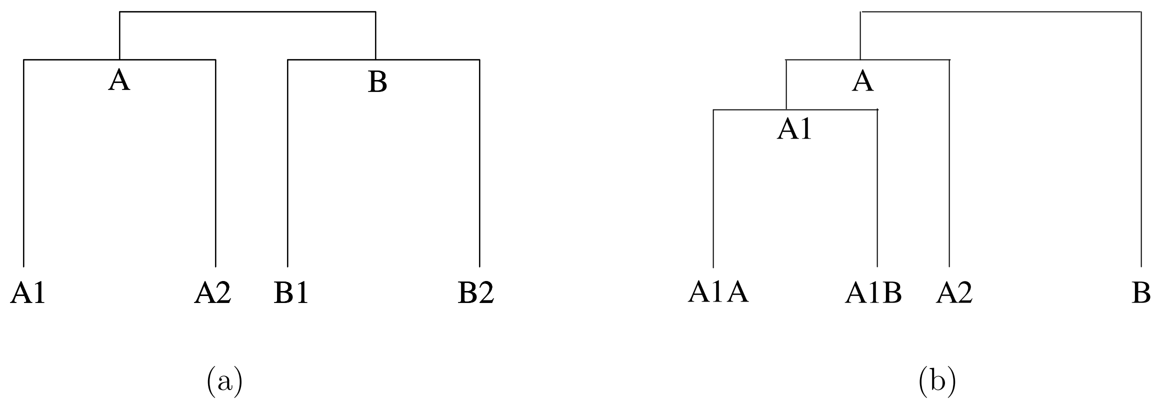


Figure 6.2: The dendrograms (or hierarchical trees) of the hierarchies shown in Figure 6.1. The leaves of the dendrogram represent the final clusters. As we move up the tree, the vertices join together to form larger and larger clusters (indicated by horizontal lines). All these clusters are joined together in a single group at the root of the tree.

least p, i.e., for all vertices $v \in S$, $\deg_S(v) \geq p \deg_{V-S}(v)$. Note that every strong defensive alliance is a p-alliance, for some $p \geq 0.5$. We define the measure of goodness of a partition of strong defensive alliances, $< A, B >$, by the maximum value $p \leq 1$, for which both A and B are p-alliances. It is easy to see that the partition $< A, B >$ of example in Figure 6.1(b) is considered better than the partition $< A, B >$ of Figure 6.1(a) by this measure. In the rest of the chapter, we present an algorithm that finds a satisfactory partition of vertices that maximizes this measure for all choices of satisfactory partitions of the given graph.

In the remainder of this chapter, we will assume the following notation.

$\mathbf{I}_n$: unit matrix of order $n \times n$.

$\mathbf{e}_n$: n dimensional vector of all ones, i.e., $\mathbf{e}_n = [\, 1 \;\; 1 \;\; \ldots \;\; 1 \,]^T$.

$\mathbf{A}(i,j)$: element of ith row and jth column in the matrix $\mathbf{A}$, $\mathbf{A}(i,j) = \mathbf{A}_{ij}$

tr$(\mathbf{A})$: trace of matrix $\mathbf{A}$, i.e. sum of diagonal elements of matrix $\mathbf{A}$.

$\mathbf{A} \bullet \mathbf{B}$: matrix inner product, $\mathbf{A} \bullet \mathbf{B} = \mathrm{tr}\left(\mathbf{A}^T\mathbf{B}\right)$

S_n: set of all symmetric $n \times n$ matrices.

M_n: set of all $n \times n$ matrices.

S_n^+: set of all positive semidefinite matrices. $\mathbf{A} \in S_n^+ \Leftrightarrow \mathbf{A} \succeq 0$

Diag$(\mathbf{x})$: diagonal matrix with vector $\mathbf{x}$ as its diagonal.

diag$(\mathbf{A})$: diagonal vector of the matrix $\mathbf{A}$.

$\mathbf{W}$: weight matrix of a weighted graph G. $\mathbf{W}_{ij}$ is the weight of the edge between vertices v_i and v_j.

The chapter is organized as follows: In Section 6.2, we review some existing graph theoretical clustering techniques. In Section 6.3, we present the details of the proposed algorithm. In Section 6.4, we show the performance of the proposed algorithm on standard data sets. Section 6.5 concludes the chapter.

6.2 Graph Theoretical Techniques for Clustering

The use of graph theoretical formalism to perform data clustering dates back to the 1950s. The concept of a minimum spanning tree was initially used in clustering in a biologically oriented method called Wrozlaw taxonomy [MM98]. Zahn [Zah71] also presented a clustering method based on the minimum spanning tree (MST) of the dissimilarity graph (as opposed to similarity graph). Urquhart [Urq82] used a similar approach along with normalization of edge weight with respect to small neighborhood. Several other graph structures such as cycles [SB94, Jac96, JI01, HW91], cliques [SB94] and shortest paths [SB94, CM98] have also been used for finding clusters in data.

The simplest cluster definitions are formulated in terms of the threshold graphs $G_t(V_t, E_t)$, where $V_t = V$ and $E_t = \{e \in E | w(e) \geq t\}$. These clusters correspond to cliques, or components of threshold graphs G_t. In 1959, Kuhn [Kuh59] defined the maximal complete subgraph of a graph as a cluster. The same definition was applied in several of the classical clustering algorithms [GK68] including *the complete-link algorithm*[JD88]. The definition is seen as the strictest definition of a cluster [AM70, RY81] and its variants are still used in some clustering applications [BY99]. Matula [Mat72] defined cluster as a maximal $k-$edge connected subgraph, which he called $k-$components of graphs. A $k-$component is a maximal induced subgraph G[S] with the property that for every partition $\langle S_1, S_2 \rangle$ of S, at least k edges of $G[S]$ are each incident with a vertex of S_1 and of S_2. In [Bri02], a similar concept is applied to define communities in a web-graph (internet-graph). In [Mat77], two other structures namely *k-bond* and *k-block* were proposed. A $k-$bond is a maximal connected induced subgraph $G[S]$ where each vertex of S has degree at least k in the subgraph $G[S]$. The problem of bi-partitioning of vertex set into sets constrained by their minimum degrees is addressed in [Sti96, Tho83, Diw00, Kan98, Haj83]. A $k-$block is a maximal induced subgraph $G[S]$ with $|S| \geq k + 1$ where $G[S']$ is connected for any $S' \subset S$ such that $|S'| \geq |S| - k + 1$.

A *clump cluster* is a set $S \subseteq V$ such that, for every $u, v \in S$ and $a, b \in V - S$, $w(u,v) > w(a,b)$. A *strong cluster* [Apr66, DF94] is a set $S \subseteq V$ such that, for every $u, v \in S$ and $a \in V - S$, $w(u,v) > \max(w(a,u), w(a,v))$. It can be seen that both clump clusters and strong clusters form hierarchies of clusters. A weaker form of the above condition defines *weak clusters*, where a set $S \subseteq V$ is called a weak cluster if for all $u, v \in S$ and $a \in V - S$, $w(u,v) > \min(w(a,u), w(a,v))$ [BD89]. Weak clusters form a weak hierarchy, i.e., for any weak clusters S_1, S_2, S_3, $S_1 \cap S_2 \cap S_3 \in \{\emptyset, S_1 \cap S_2, S_2 \cap S_3, S_3 \cap S_1\}$. A $\pi-$*cluster* $S \subset V$ is defined by the condition that $d(S) \geq \pi$, where, $d(S) = \sum_{u,v \in S} \frac{w(u,v)}{|S|^2}$ is the average similarity within S. A *strict cluster* is a set $S \subset V$ such that for any $a \in V - S$ and $v \in S$, $w(a,S) < \frac{w(S)}{2} \leq w(v,S)$, where w(a,S) is the average similarity of a and S and $w(S)$ is the average similarity of S.

The clumps and component clusters can be found by finding cliques and components in graphs, while the other concepts are not as well developed.

Genkin and Muchnik [Gen93] defined the concept of $t-$*clusters*, where a set $S \subseteq V$ is a $t-$*cluster* or is a $t-$*stable set* if and only if the following conditions hold:

$$\pi(x,S) \geq t \text{ for all } x \in S$$

$$\pi(x,S) < t \text{ for all } x \in V - S$$

where $\pi(x,S)$ is the measure of similarity of data item x with respect to set S. It is shown that many previously defined concepts of clusters, for example, cliques, $k-$components, and $k-$blocks can be modelled in terms of $t-$clusters for different choices for π. The algorithms for finding $t-$clusters also depends on this choice.

Recently, Pavan and Pelillo [PP03] proposed a generalization of the concept of maximal clique to edge-weighted graphs based on the study of a continuous formulation of the maximum clique problem by Motzkin and Straus [MS65]. They termed the proposed structure as *dominant set*. Let $S \subseteq V$ be a non-empty subset of vertices and $v \in V$. The *average weighted degree* of v w.r.t. S is defined as:

$$\mathrm{awdeg}_S(v) = \frac{1}{|S|} \sum_{u \in S} w(u, v).$$

In addition, if $u \notin S$, we define:

$$\phi_S(u, v) = w(u, v) - \mathrm{awdeg}_S(v).$$

Intuitively, $\phi_S(u, v)$ measures the similarity between vertices u and v, with respect to the average similarity between vertex v and its neighbors in S. The weight of vertex v w.r.t. S is

$$w_S(v) = \begin{cases} 1, & \text{if } |S| = 1 \\ \sum_{u \in S - \{v\}} \phi_{S-\{v\}}(u, v) w_{S-\{v\}}(u), & \text{otherwise} \end{cases}$$

Moreover, the total weight of set S is defined as:

$$W(S) = \sum_{v \in S} w_S(v).$$

A non empty subset of vertices $S \subseteq V$ such that $W(T) > 0$ for any non-empty $T \subseteq S$ is said to be dominant if:

Q.1. $w_S(v) > 0$, for all $v \in S$,

Q.2. $w_{S \cup \{v\}}(v) < 0$, for all $v \notin S$.

The first condition corresponds to internal homogeneity of cluster while the second models the external inhomogeneity. A quadratic programming approach was used to find dominant sets in graphs.

The methods described above, first define a notion of what a cluster is, and then seek a partition of the given data into the defined structure. Apart from these methods, various kinds of graph partitioning problems define the clustering criterion without explicitly defining a notion of cluster. Given a graph G, a partition $P = \langle V_1, V_2, \ldots, V_k \rangle$ can be found by minimizing or maximizing some global criterion. For example, maximizing within cluster similarity:

$$f(P) = \sum_{i=1}^{k} \sum_{u,v \in V_i} \frac{w(u,v)}{|V_i|} = \sum_{i=1}^{k} f(V_k).$$

Another possibility to model this problem is using minimum $k-$cuts, where a minimum $k-$cut problem or k-way Split problem [GH88, HS85, SV91] is defined as follows: Given an edge weighted graph $G = (V, E)$ and an integer k, find a minimum weight set of edges $E' \subseteq E$ whose removal separates the graph into at least k nonempty connected components. The problem is NP-hard for arbitrary k, while a polynomial algorithm exists for each fixed $k > 2$, even for arbitrary graphs [GH88].

Another similar problem is called $k-$way cut or Multiway Cuts [DJP94], which is defined as follows: Given an edge weighted graph $G = (V, E)$, a set $S = \{s_1, s_2, \ldots, s_k\}$ of k specified vertices or *terminals*, find a minimum weight set of edges $E' \subseteq E$ such that the removal of E' from E disconnects each terminal from all the others. The problem is NP-Hard even for

$k = 3$; however it can be solved in polynomial time for planar graphs for any fixed k. The planar problem is also NP-Hard for arbitrary k.

Due to the computational cost of the $k-$way partitioning problem, it is a usual practice to approximate the general $k-$way partitioning solution by recursive bi-partitioning, where, at each step, the graph is partitioned into two sets based on the partitioning criterion (though a few exceptions exist, see, for example, [GWW01]). Finding a minimum 2-way cut or simply minimum edge cutset is a polynomial problem [GH61]. As a matter of fact, all minimum cuts in a graph can be generated in polynomial time [NNI97]. Wu and Leahy [WL93] proposed a clustering method based on minimum edge cutset of graphs. The problem of finding $k-$way partition was then approximated by recursively finding the minimum cuts that bisect the existing clusters.

Though the minimum cut algorithm bi-partitions the graph in the optimal way, it has the tendency to create very small clusters (i.e., the partitions formed are unbalanced in terms of sizes of the sets). However, imposing constraints on the sizes of sets in the partition makes the problem NP-Hard [WW91].

Shi and Malik [SM00] proposed the normalized cut measure for clustering, where the *normalized cut* partitioning the vertex set V into sets A and B is defined as follows:

$$Ncut(A, B) = \frac{\sum_{u \in A, v \in B} w(u, v)}{\sum_{u \in A, t \in V} w(u, t)} + \frac{\sum_{u \in A, v \in B} w(u, v)}{\sum_{u \in B, t \in V} w(u, t)}$$

The problem is then to find a minimum normalized cut of graph G. The problem of finding a minimum normalized cut is NP-Hard and an approximate algorithm known as spectral clustering is applied to find the solution. Perona and Freeman [PF98] proposed an asymmetric variation of normalized cut, which they referred to as *foreground cut*. Define one of the two subsets of G to be a foreground F and its complement $B = V - F$ to be the background, then the foreground cut $N(F)$ is given as:

$$N(F) = \frac{\sum_{u \in F, v \in B} w(u,v)}{\sum_{u \in F, t \in F} w(u,t)}$$

Once again, a minimum foreground cut is sought in the graph. Sarkar and Soundararajan [SS00] also used a graph partition based framework and normalized the edge cutset by the product of the sizes of each partition and referred to as *average cut*. Both of these problems are again NP-Hard and can be approximated by spectral clustering.

Similar ideas of partitioning the vertex set of graphs using some global criterion are employed in [Vek00, Wan01, KVV00, IG98, Wei99, YS01].

6.3 Clustering Using Maximum Satisfactory Minimum Cut

6.3.1 Problem Definition

In this section, we present the details of the proposed algorithms. As mentioned in Section 6.1, that given a graph $G = (V, E)$, we seek a bipartition of the vertex set V into sets A, B, such that both A and B are defensive p-alliances and p is maximum among all such bipartitions. In the case, when there is more than one such partition, we find one which has the minimum number of edges between the sets A and B. We refer to such a cut as a *maximum satisfactory minimum cut* of the graph G. Formally, we define the problem as follows:

MAXIMUM SATISFACTORY MINIMUM CUT (MSMC)

Input: A Graph $G(V, E)$ and a weight function $w : E \rightarrow \mathbb{R}$.

Question: Find a partition A,B of V and a real number p, $0 \leq p \leq 1$, such that,

i. for all $v \in A$, $\sum_{u \in N(v) \cap A} w(u, v) \geq p \sum_{u \in N(v)} w(u, v)$,

ii. for all $v \in B$, $\sum_{u \in N(v) \cap B} w(u, v) \geq p \sum_{u \in N(v)} w(u, v)$,

iii. p is maximum among all partitions, A, B, satisfying (i) and (ii).

iv. The cut between sets A and B is minimum among all the partitions satisfying (i),(ii) and (iii).

The above problem is NP-Hard (By reduction from Satisfactory Partition). In order to find an approximate solution of the above problem, we start by first presenting a quadratic programming formulation of MSMC. Let $V = \{v_1, v_2, \ldots, v_n\}$. For each vertex $v_i \in V$, we define a binary variable x_i. We want to find a partition, A, B, of V into defensive p-alliances, $A = \{v_i | x_i = 0\}$ and $B = \{v_i | x_i = 1\}$ such that p is maximum among all such partitions. A partition A, B of V is a satisfactory p-partition (a partition into defensive p-alliances, if and only if for every vertex v_i,

$$-\sum_{v_j \in N(v_i)} w(v_i, v_j) \leq p(2x_i - 1) \sum_{v_j \in N(v_i)} w(v_i, v_j) - \sum_{v_j \in N(v_i)} x_j w(v_i, v_j) \leq 0$$

We also want both sets A and B to be nonempty. As a matter of fact, for the sets A and B to be strong defensive alliances, each of them must have at least 2 vertices. Thus, we must have $\sum_{i=1}^{n} x_i \geq 2$ and $\sum_{i=1}^{n} x_i \leq n - 2$.

The complete quadratic program can now be written as follows:

Maximize: $Kp - \sum_{1 \leq i < j \leq n} w(v_i, v_j)(x_i - x_j)^2$

Subject to the following constraints:

- $p(2x_i - 1) \sum_{v_j \in N(v_i)} w(v_i, v_j) - \sum_{v_j \in N(v_i)} x_j w(v_i, v_j) \leq 0,\ 1 \leq i \leq n$
- $p(2x_i - 1) \sum_{v_j \in N(v_i)} w(v_i, v_j) - \sum_{v_j \in N(v_i)} x_j w(v_i, v_j) \geq -\sum_{v_j \in N(v_i)} w(v_i, v_j),\ 1 \leq i \leq n$
- $x_i^2 = x_i,\ 1 \leq i \leq n$
- $\sum_{i=1}^{n} x_i \geq 2$
- $\sum_{i=1}^{n} x_i \leq n - 2$
- $p \geq 0$
- $p \leq 1$

The second term, $\sum_{1 \leq i < j \leq n} w(v_i, v_j)(x_i - x_j)^2$ in the objective function is the value of the cut and minimizing it corresponds to maximizing the objective function. On the other hand, the increase in the value of p increases the term Kp and hence, the objective function. Here, K is a constant that controls the precision of calculating p and should be chosen such that the increase in the value of p by the amount of required precision has more effect on the value of objective function than the value of any of the cut. In general, $K \gg \sum_{1 \leq i < j \leq n} w(v_i, v_j)$.

The functional and constraints of the above quadratic program have the form of a general quadratic functional, i.e., $\mathbf{x}^T \mathbf{Q} \mathbf{x} + 2\mathbf{b}^T \mathbf{x}$. We can homogenize this program by introducing a new variable x_0 and setting it to 1. We may then replace each linear term ax_i by the quadratic term $ax_i x_0$. Thus, the homogenous version of the above quadratic program can be written as:

Maximize: $Kpx_0 - \sum_{1 \leq i < j \leq n} w(v_i, v_j)(x_i - x_j)^2$

Subject to the following constraints:

- $p(2x_i - x_0) \sum_{v_j \in N(v_i)} w(v_i, v_j) - \sum_{v_j \in N(v_i)} x_j x_0 w(v_i, v_j) \leq 0, \ 1 \leq i \leq n$
- $p(2x_i - x_0) \sum_{v_j \in N(v_i)} w(v_i, v_j) - \sum_{v_j \in N(v_i)} x_j x_0 w(v_i, v_j) \geq - \sum_{v_j \in N(v_i)} w(v_i, v_j), \ 1 \leq i \leq n$
- $x_i^2 = x_i x_0, \ 1 \leq i \leq n$
- $\sum_{i=1}^{n} x_i x_0 \geq 2$
- $\sum_{i=1}^{n} x_i x_0 \leq n - 2$
- $px_0 \geq 0$
- $px_0 \leq 1$
- $x_0^2 = 1$

6.3.2 Semidefinite Relaxation of MSMC

The quadratic program given in the previous subsection is equivalent to MSMC problem, and hence, is NP-Hard. We now present a semidefinite relaxation of this quadratic program. Semidefinite programming is linear programming over the cone of semidefinite matrices[WSV00, LS91]. In comparison to standard linear programming the vector $\mathbf{x} \in \mathbb{R}^n$ of variables is replaced by a matrix variable $\mathbf{X} \in S_n^+$, where S_n^+ is the set of all $n \times n$ positive semidefinite matrices. In other words, the cone of the nonnegative orthant $x \geq 0$ is replaced by the cone of semidefinite matrices $\mathbf{X} \succeq 0$. A semidefinite program is an optimization problem of the following form:

Maximize $\mathbf{A}_0 \bullet \mathbf{X}$

subject to $\mathbf{A}_i \bullet \mathbf{X} = c_i$, $1 \leq i \leq m$, $\mathbf{X} \succeq 0$

where $\mathbf{X} \in M_n$ and for all $0 \leq i \leq m$, $\mathbf{A}_i \in M_n$.

The trick to obtain a polynomially solvable relaxation of the quadratic programming problems (as defined in the previous subsection), is to think of each variable x_i as vectors in a higher dimensional space $\mathbb{R}^k$, and subsequently, multiplications of two vectors as inner product in this space [Hel00]. Different choices of the dimension $k < n$ (where n is the number of variables of the quadratic program of concern) of this new space provides different problems of geometric nature, but usually are still NP-Hard. However, if we chose the dimension of the new space to be equal to the number of original variables then we get a polynomial time solvable problem. Using this technique, we may write a polynomially solvable relaxation of the quadratic program of previous subsection as follows:

Maximize: $K\mathbf{p}^T\mathbf{x}_0 - \sum_{1\le i<j\le n} w(v_i,v_j)(\mathbf{x}_i-\mathbf{x}_j)^T(\mathbf{x}_i-\mathbf{x}_j)$

- $\mathbf{p}^T(2\mathbf{x}_i-\mathbf{x}_0)\sum_{v_j\in N(v_i)} w(v_i,v_j) - \sum_{v_j\in N(v_i)} \mathbf{x}_j^T\mathbf{x}_0 w(v_i,v_j) \le 0,\ 1\le i\le n$
- $\mathbf{p}^T(2\mathbf{x}_i-\mathbf{x}_0)\sum_{v_j\in N(v_i)} w(v_i,v_j) - \sum_{v_j\in N(v_i)} \mathbf{x}_j^T\mathbf{x}_0 w(v_i,v_j) \ge -\sum_{v_j\in N(v_i)} w(v_i,v_j),\ 1\le i\le n$
- $\mathbf{x}_i^T\mathbf{x}_i = \mathbf{x}_i^T\mathbf{x}_0,\ 1\le i\le n$
- $\sum_{i=1}^n \mathbf{x}_i^T\mathbf{x}_0 \ge 2$
- $\sum_{i=1}^n \mathbf{x}_i^T\mathbf{x}_0 \le n-2$
- $\mathbf{p}^T\mathbf{x}_0 \ge 0$
- $\mathbf{p}^T\mathbf{x}_0 \le 1$
- $\mathbf{x}_0^T\mathbf{x}_0 = 1$

In order to derive a semidefinite relaxation of the above homogenous quadratic program of the form $\mathbf{x}^T\mathbf{Q}\mathbf{x}$, we note the fact that $\mathbf{x}^T\mathbf{Q}\mathbf{x} = \mathbf{Q}\bullet\mathbf{x}\mathbf{x}^T$ and replace $\mathbf{x}\mathbf{x}^T$ with a matrix $\mathbf{X}$ to come up with a linear objective function. The variables of this new system are the elements of matrix $\mathbf{X}$. Imposing the condition that $\mathbf{X}\succeq 0$, i.e., $\mathbf{X}$ is a semidefinite matrix gives us a semidefinite program. The inequalities in the constraints are replaced with equalities by introducing slack variables (one for each inequality). These steps yield the following semidefinite program for the MSMC problem that consists of $3n+6$ variables (including $2n+4$ slack variables) and $3n+6$ constraints.

Maximize $\mathbf{A}_0 \bullet \mathbf{X}$

$\mathbf{X} = \begin{bmatrix} \mathbf{X_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} \end{bmatrix} \in S_{3n+6}$, where $\mathbf{X_1} = \mathbf{x}\mathbf{x}^T$, $\mathbf{x} = [\ x_1 \ \ x_2 \ \ \ldots \ \ x_n \ \ p \ \ x_0\ ]^T$ and $\mathbf{L}$ is a diagonal matrix whose diagonal is the vector of $2n{+}4$ slack variables, i.e., $\mathbf{L} = \text{Diag}\left([\ l_1 \ \ l_2 \ \ \ldots \ \ l_{2n+4}\ ]^T\right)$.

$\mathbf{A}_0 = \begin{bmatrix} \mathbf{B}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$, where $\mathbf{B}_1 = \mathbf{W} - \text{Diag}\left(\mathbf{W}\mathbf{e_n}\right)$, and $\mathbf{B}_2 = \begin{bmatrix} 0 & \frac{K}{2} \\ \frac{K}{2} & 0 \end{bmatrix}$.

Constraints:

- $\mathbf{X} \succeq 0$
- $\mathbf{A}_i \bullet \mathbf{X} = 0, \ 1 \leq i \leq n.$
  - i. $\mathbf{A}_i(i, n+1) = \mathbf{A}_i(n+1, i) = \sum_{v_j \in N(v_i)} w(v_i, v_j)$
  - ii. for all $j \neq i$, $1 \leq j \leq n$, $\mathbf{A}_i(j, n+2) = \mathbf{A}_i(n+2, j) = -\frac{1}{2} w(v_i, v_j)$,
  - iii. $\mathbf{A}_i(n+1, n+2) = \mathbf{A}_i(n+2, n+1) = -\frac{1}{2} \sum_{v_j \in N(v_i)} w(v_i, v_j)$, and
  - iv. $\mathbf{A}_i(n+i+2, n+i+2) = 1$.
- $\mathbf{A}_i \bullet \mathbf{X} = -\sum_{v_j \in N(v_{i-n})} w(v_{i-n}, v_j), \ n+1 \leq i \leq 2n.$
  - i. $\mathbf{A}_i(i-n, n+1) = \mathbf{A}_i(n+1, i-n) = \sum_{v_j \in N(v_{i-n})} w(v_{i-n}, v_j)$
  - ii. for all $j \neq (i-n)$, $1 \leq j \leq n$, $\mathbf{A}_i(j, n+2) = \mathbf{A}_i(n+2, j) = -\frac{1}{2} w(v_{i-n}, v_j)$,

iii. $\mathbf{A}_i(n+1, n+2) = \mathbf{A}_i(n+2, n+1) = -\frac{1}{2}\sum_{v_j \in N(v_{i-n})} w(v_{i-n}, v_j)$, and

iv. $\mathbf{A}_i(2n+i+2, 2n+i+2) = -1$.

- $\mathbf{A}_i \bullet \mathbf{X} = 0,\ \ 2n+1 \leq i \leq 3n$.

  i. $\mathbf{A}_i(i-2n, i-2n) = 1$ and

  ii. $\mathbf{A}_i(i-2n, n+2) = \mathbf{A}_i(n+2, i-2n) = -\frac{1}{2}$.

- $\mathbf{A}_{3n+1} \bullet \mathbf{X} = 2$.

  i. for all j, $1 \leq j \leq n$, $\mathbf{A}_{3n+1}(j, n+2) = \mathbf{A}_{3n+1}(n+2, j) = \frac{1}{2}$ and

  ii. $\mathbf{A}_{3n+1}(3n+3, 3n+3) = -1$.

- $\mathbf{A}_{3n+2} \bullet \mathbf{X} = n-2$.

  i. for all j, $1 \leq j \leq n$, $\mathbf{A}_{3n+2}(j, n+2) = \mathbf{A}_{3n+2}(n+2, j) = \frac{1}{2}$ and

  ii. $\mathbf{A}_{3n+2}(3n+4, 3n+4) = 1$.

- $\mathbf{A}_{3n+3} \bullet \mathbf{X} = 0$.

  i. $\mathbf{A}_{3n+3}(n+1, n+2) = \mathbf{A}_{3n+3}(n+2, n+1) = \frac{1}{2}$ and

  ii. $\mathbf{A}_{3n+3}(3n+5, 3n+5) = -1$.

- $\mathbf{A}_{3n+4} \bullet \mathbf{X} = 1$.

  i. $\mathbf{A}_{3n+4}(n+1, n+2) = \mathbf{A}_{3n+4}(n+2, n+1) = \frac{1}{2}$ and

  ii. $\mathbf{A}_{3n+4}(3n+6, 3n+6) = 1$.

- $\mathbf{A}_{3n+5} \bullet \mathbf{X} = 1$. $\mathbf{A}_{3n+5}(n+2, n+2) = 1$.

The feasible matrix $\mathbf{X} = \begin{bmatrix} \mathbf{X_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} \end{bmatrix}$ of the semidefinite program can be interpreted as the Gram matrix of vectors $\mathbf{v}_i \in \mathbb{R}^n$, $1 \leq i \leq 3n+6$, which correspond to the higher dimensional relaxation of variables of the original quadratic program. For any factorization of a feasible $\mathbf{X}$ into $\mathbf{V}^T\mathbf{V}$ with $\mathbf{V} \in M_{3n+6}$, the columns of $\mathbf{V}$ yields such vectors $\mathbf{v}_i$. Such a factorization can be obtained by using eigenvalue decomposition as follows. Let $\mathbf{X} = \mathbf{PDP}^T$ be the eigenvalue decomposition of $\mathbf{X}$, then $\mathbf{X} = \mathbf{PD}^{\frac{1}{2}}\mathbf{D}^{\frac{1}{2}}\mathbf{P}^T = \mathbf{V}^T\mathbf{V}$, where $\mathbf{V} = \mathbf{D}^{\frac{1}{2}}\mathbf{P}^T$. Thus a solution of the higher dimensional relaxation of the original quadratic program for MSMC can be obtained in polynomial time by solving the above semidefinite program. These vectors can then be partitioned into two groups (0 and 1), based on the values of their norms. We use a method similar to [SM00] to find this partition of the relaxed variables.

In order to solve the semidefinite program, a number of iterative interior point algorithms have been suggested in the literature. Details of these algorithms can be found in [NN94, Ye97]. For the purposes of this dissertation, we have used a version of interior point algorithms, known as primal dual interior point algorithm [Stu97]. The implementation [YFK03] is based on Mahrotra predictor corrector infeasible primal dual interior point algorithm [Meh92].

6.4 Results

6.4.1 Zachary's Karate Club Network

Wayne Zachary [Zac77] observed social interactions between the members of a karate club at an American university in the early 1970s for two years. The data consists of the ties between members of the club based on their social interactions with each other and is shown in Figure 6.3[1]. He used these data and an information flow model of network conflict resolution to explain the split-up of this group following disputes among the members. The split-up of the group occurred after the dispute between the administrator and the principal karate teacher. In Figure 6.3, we show the partition obtained by our algorithm for this network. The administrator and the instructor are represented by vertices 1 and 33 respectively. The vertices in each cluster are shown using different color. From ground truth, we find that only vertex 3 is incorrectly classified by our algorithm, the rest of the split found by the algorithm is consistent with the actual split of the club. Also, from the figure it is clear that the vertex 3 has equal number of ties with both clusters and its assignment to any cluster cannot be justified by the given data and hence, must be made arbitrarily. The final clustering of the data is shown in Figure 6.4, where a total of 3 clusters were detected.

[1] The networks shown in this chapter are drawn using Pajek[BM98]

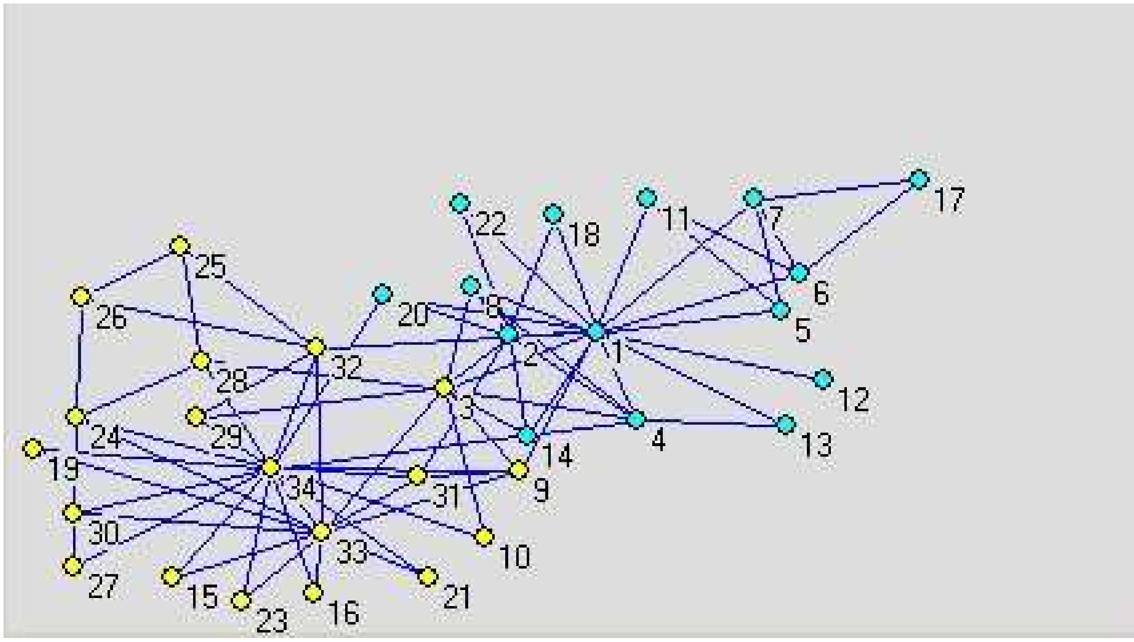


Figure 6.3: The network of ties between the members of the karate club from Zachary Karate Club data set. The bipartition of the data generated by our algorithm is shown by using different colors for the members belonging to different clusters. The members with greater ties with the administrator (vertex 1) are colored blue whereas the members with greater ties with the instructor (vertex 33) are colored yellow. Only the coloring of vertex 3 is inconsistent with the actual split of the club.

6.4.2 Zoo Database

Zoo Database was created by David Forsyth (PC/BEAGLE User's Guide) and contains the data of 101 animals. The entry for each animal consists of 15 binary attributes, such as, whether or not the animal has fins, feathers, hairs, etc, or if the animal lays eggs, or is

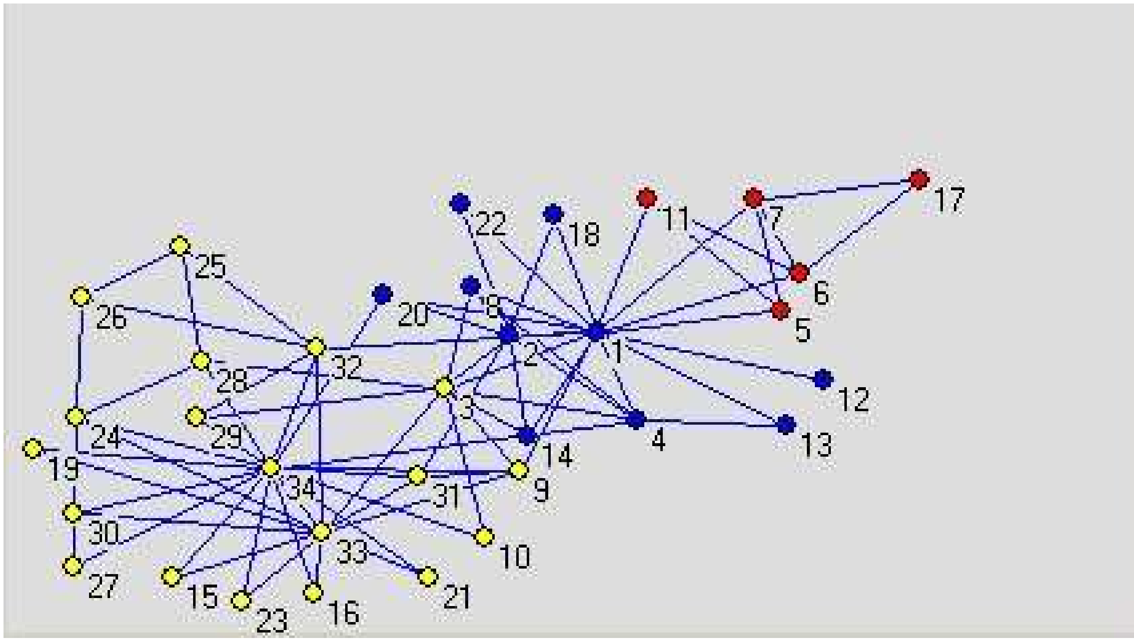


Figure 6.4: Final clustering of the Zachary Karate Club data generated by the proposed algorithm.

venomous. There is also one numeric attribute that provides the number of legs for each animal. Some properties of the database are listed in Table 6.1.

We defined the dissimilarity between two animals as the hamming distance between their binary feature vector (Though legs is a numeric feature, it only takes five discrete values and can be converted into as many binary features). The edge weights of the graph were then defined as the number of binary features minus the hamming distance between the two feature vectors. This graph was used as an input to our algorithm. Our algorithm found 11 groups in the data (this happened because some of the groups in the data set had the tendency for further subdivision and is not considered an error on the part of algorithm). The groups obtained by the algorithm are shown in Figure 6.5. The same information is also presented in tabular form in Table 6.2. The class hierarchy is shown in the form of dendrogram in Figure 6.6. From these results, it can be clearly seen that the clusters generated by the algorithm are quite consistent with the actual classification of Table 6.1.

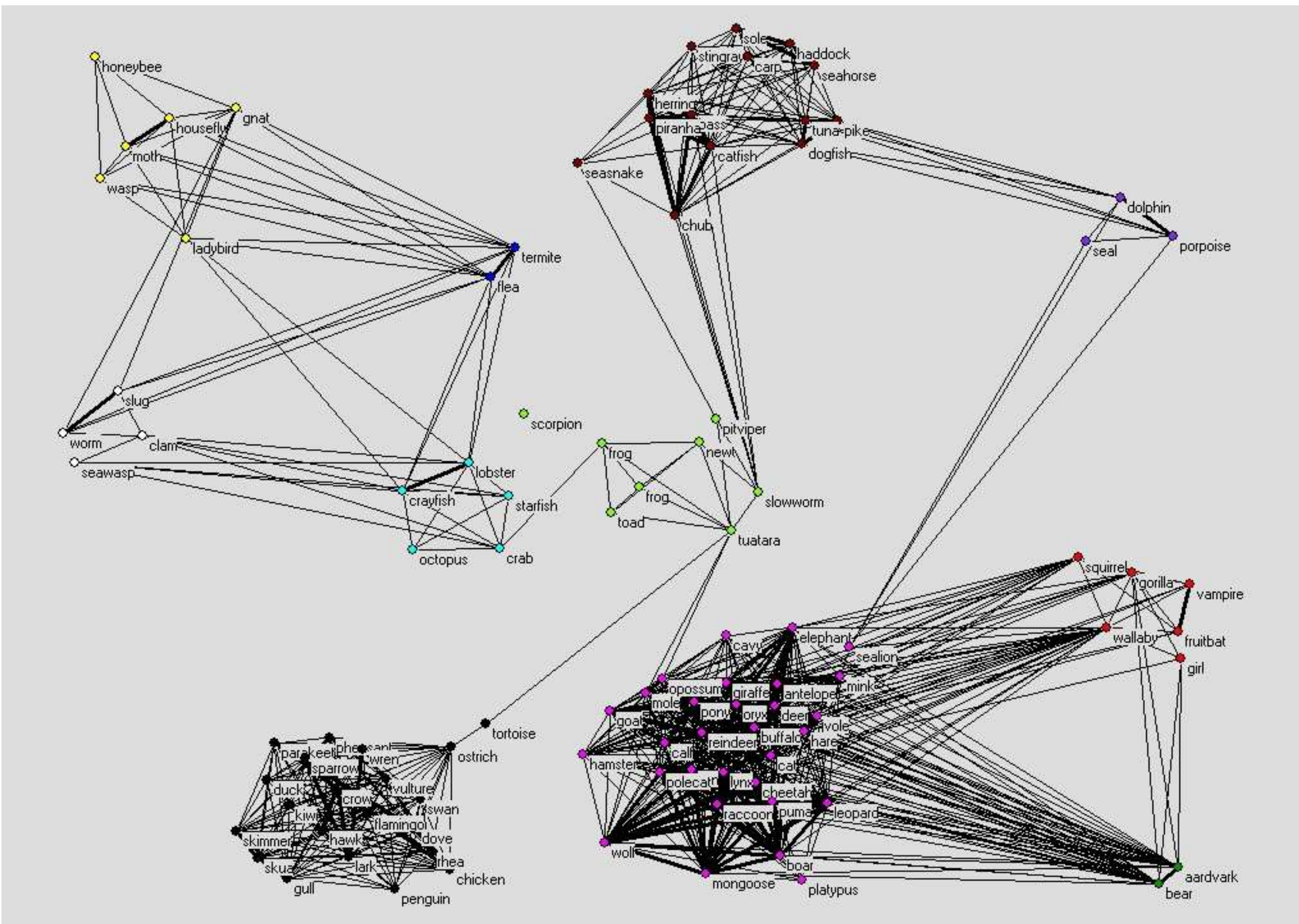


Figure 6.5: Grouping among the animals of zoo database. A total of 9 groups were recognized by the algorithm.

Table 6.1: General Information about Zoo database.

Zoo Database	
Number of Classes	7
Number of Instances	101
Number of Attributes	16
Attribute Information (Type)	Hair (Boolean), Feathers(Boolean), Eggs(Boolean), Milk (Boolean), Airborne (Boolean), Aquatic (Boolean), Predator (Boolean),Fins (Boolean), Legs (Numeric), Backbone (Boolean),Breathes (Boolean), Toothed (Boolean), Tail (Boolean), Venomous (Boolean), Domestic (Boolean), Catsize (Binary)
Class Distribution	
Class 1 (41 Animals)	aardvark, antelope, bear, boar, buffalo, calf, cavy, cheetah, deer, dolphin, elephant, fruitbat, giraffe, girl, goat, gorilla, hamster, hare, leopard, lion, lynx, mink, mole, mongoose, opossum, oryx, platypus, polecat, pony, porpoise, puma, pussycat, raccoon, reindeer, seal, sea lion, squirrel, vampire, vole, wallaby,wolf.
Class 2 (20 Animals)	chicken, crow, dove, duck, flamingo, gull, hawk, kiwi lark, ostrich, parakeet, penguin, pheasant, rhea, skimmer, skua, sparrow, swan, vulture, wren.
Class 3 (5 Animals)	pitviper, seasnake, slowworm, tortoise, tuatara.
Class 4 (13 Animals)	bass, carp, catfish, chub, dogfish, haddock, herring, pike, piranha, seahorse, sole, stingray, tuna.
Class 5 (4 Animals)	frog, frog, newt, toad.
Class 6 (8 Animals)	flea, gnat, honeybee, housefly, ladybird, moth, termite, wasp.
Class 7 (10 Animals)	clam, crab, crayfish, lobster, octopus, scorpion, sea wasp, slug, starfish, worm.

Table 6.2: Clusters of animals in the Zoo database as found by the proposed algorithm.

Number of Classes	11
Class Distribution	
Class 1 (2 Animals)	aardvark, bear.
Class 2 (30 Animals)	antelope, boar, buffalo, calf, cavy, cheetah, deer, elephant, giraffe, goat, hamster, hare, leopard, lion, lynx, mink, mole, mongoose, opossum, oryx, platypus, polecat, pony, puma, cat, raccoon, reindeer, sea lion, vole, wolf.
Class 3 (6 Animals)	fruitbat, girl, gorilla, squirrel, vampire, wallaby.
Class 4 (3 Animals)	dolphin, porpoise, seal
Class 5 (14 Animals)	bass, carp, catfish, chub, dogfish, haddock, herring, pike, piranha, seahorse, sea snake, sole, stingray, tuna.
Class 6 (21 Animals)	chicken, crow, dove, duck, flamingo, gull, hawk, kiwi lark, ostrich, parakeet, penguin, pheasant, rhea, skimmer, skua, sparrow, swan, tortoise, vulture, wren.
Class 7 (8 Animals)	frog, frog, newt, pitviper, scorpion, slowworm, toad, tuatara.
Class 8 (2 Animals)	flea,termite.
Class 9 (6 Animals)	gnat, honeybee, housefly, ladybird, moth, wasp
Class 10 (5 Animals)	crab, crayfish, lobster, octopus, starfish.
Class 11 (4 Animals)	clam, sea wasp, slug, worm.

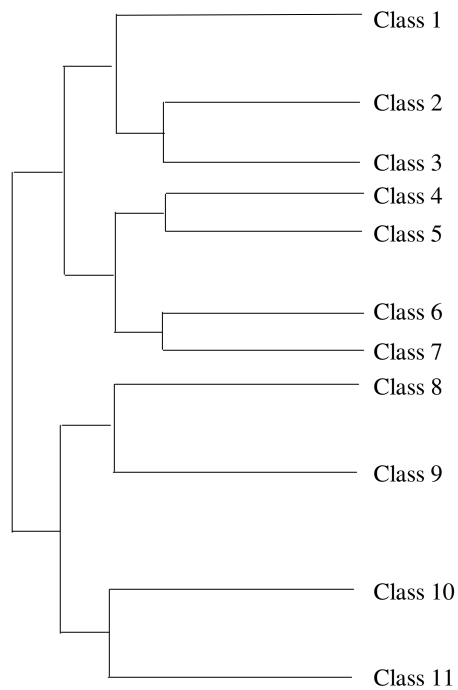


Figure 6.6: Dendrogram of the clusters obtained by the proposed algorithm.

6.4.3 Networks of Fictional Characters

In this subsection, we present results on two data sets based on the ties between the fictional characters. The first data set provides the ties between different characters of Victor Hugo's novel *Les Miserables*, whereas the other provides the same for Mark Twain's Huckleberry Finn. Compiled by Knuth[Knu93] as part of the Stanford GraphBase project, the data sets define the ties between two characters based on whether or not the two ever appeared in the same scene. Our algorithm found 10 clusters of (more than one) characters of *Les Miserables*, that are shown in Figure 6.7. The clusters generated are consistent with the different subplots of the novel. We compare the clusters obtained by our algorithm to the ones achieved by Newman and Girvan [NG04], in Table 6.4.3. A total of 9 groups were found among the characters of Huckleberry Finn (Figure 6.8). The algorithm correctly recognized the family structure of Grangerfords and also the network of Phelps. The other dominant groups are centered around the main characters Tom Sawyer, slave Jim, Doctor Robinson and Duke.

6.4.4 Other Standard Data Sets

In addition to the social structure networks of previous data sets, we also tested our algorithm on five other standard data sets for classification algorithms. These data sets are Wine, Iris Plant, Hepatitis, Dermatology Database, and Protein Localization Sites. (Some facts about these databases are summarized in Tables 6.4-6.8. We also compare the performance of our algorithm on these data sets with the performance of normalized cut algorithm by Shi and Malik[SM00].

The first data set, *Wine Recognition Data* (Table 6.4) consists of the results of a chemical analysis of wines grown in the same region in Italy but derived from three different cultivars. The analysis determined the quantities of 13 constituent found in each of the three types of wines. There are a total of 178 samples of wines in the data set, 59 belonging to class 1, 71 of class 2 and 48 of class 3. When run over the complete data set, our algorithm correctly classifies all the samples of first and third class, however 9 samples of the class

Table 6.3: Grouping of characters of Victor Hugo's *Les Miserables*.

Group No.	Proposed Method	Newman and Girvan's Method[NG04]
1	Count Countess DeLo Cravatte Geborand Marquis Myriel Napoleon Old Man	Count Countess DeLo Cravatte Geborand Marquis de Champtercier Myriel Napoleon Old Man Mlle Baptistine Mme Magloire
2	Fauchlevent Gibier Mother Innocent	Fauchlevent Gibier Mother Innocent
3	Jondrette Mme Burgon	Jondrette Mme Burgon
4	Child 1 Child 2	Child 1 Child 2
5	Bahorel Bossuet Combeferre Courfeyrac Enjolras Feuilly Gavroche Grantaire Joly Mabeuf Marius Mme Hucheloup Prouvaire Mother Plutarch	Bahorel Bossuet Combeferre Courfeyrac Enjolras Feuilly Gavroche Grantaire Joly Mabeuf Marius Mme Hucheloup Prouvaire
6		Mother Plutarch

Group No.	Proposed Method	Newman and Girvan's Method[NG04]
7	Baroness T Gillenormand Lt Gillenormand Mlle Gillenormand Mlle Vaubois Mme Pontmercy	Baroness T Gillenormand Lt Gillenormand Mlle Gillenormand Mlle Vaubois Mme Pontmercy Cosette Old Woman 2 Magnon Toussaint
8	Boulatruelle Cosette Magnon Old woman 2 Toussaint Anzelma Babet Brujon Claquesous Eponine Gueulemer Javert Mme Thenardier Montparnasse Pontmercy Thenardier	Anzelma Babet Brujon Claquesous Eponine Gueulemer Javert Mme Thenardier Montparnasse Pontmercy Thenardier
9		Boulatruelle

Group No.	Proposed Method	Newman and Girvan's Method[NG04]
10	Blacheville Dahlia Fameuil Fantine Favourite Listolier Felix Tholomyes Zephine	Blacheville Dalhia Fameuil Fantine Favourite Listolier Felix Tholomyes Zephine Marguerite Perpetue
11	Mlle Baptistine Perpetue Mme Magloire Marguerite Gervais Isabeau Labarre Mme De R Old Woman 1 Scaufflaire Simplice Jean Valjean	Gervais Isabeau Labarre Mme De R Old Woman 1 Scaufflaire Simplice Jean Valjean Bamatabois Brevet Champmathieu Chenildeiu Cockepaille Judge
12	Bamatabois Brevet Champmathieu Chenildeiu Cockepaille Judge	

Table 6.4: General information about Wine Recognition database.

Data Set: Wine Recognition	
Number of Instances	178
Number of Attributes	13
Attributes	alcohol, malic acid, ash, alcalinity of ash, magnesium, total phenols, flavanoids, nonflavanoid phenols, proanthocyanins, color intensity, hue, OD280/OD315 of diluted wines, proline
Class Distribution	Class 1: 59 instances Class 2: 71 instances Class 3: 48 instances

Table 6.5: General information about Iris Plant data set

Data Set: Iris Plant	
Number of Instances	150
Number of Attributes	4
Attributes	sepal length, sepal width, petal length, petal width
Class Distribution	Class 1 (Iris Setosa): 50 instances Class 2 (Iris Versicolour): 50 instances Class 3 (Iris Virginica): 50 instances

Table 6.6: General information about Hepatitis data set

Data Set: Hepatitis	
Number of Instances	155
Number of Attributes	19
Attributes	age, sex, steroid, antivirals, fatigue, malaise, anorexia, liver big, liver firm, spleen palpable, spiders, ascites, varices, bilirubin, alk phosphate, sgot, albumin, protime, histology
Class Distribution	Class 1 (Live): 123 instances Class 2 (Die) : 32 instances

Table 6.7: General information about Dermatology data set

Data Set: Dermatology	
Number of Instances	366
Number of Attributes	34
Class Distribution	Class 1 (psoriasis): 112 instances Class 2 (seboreic dermatitis): 61 instances Class 3 (lichen planus): 72 instances Class 4 (pityriasis rosea): 49 instances Class 5 (cronic dermatitis): 52 instances Class 6 (pityriasis rubra pilaris) : 20 instances

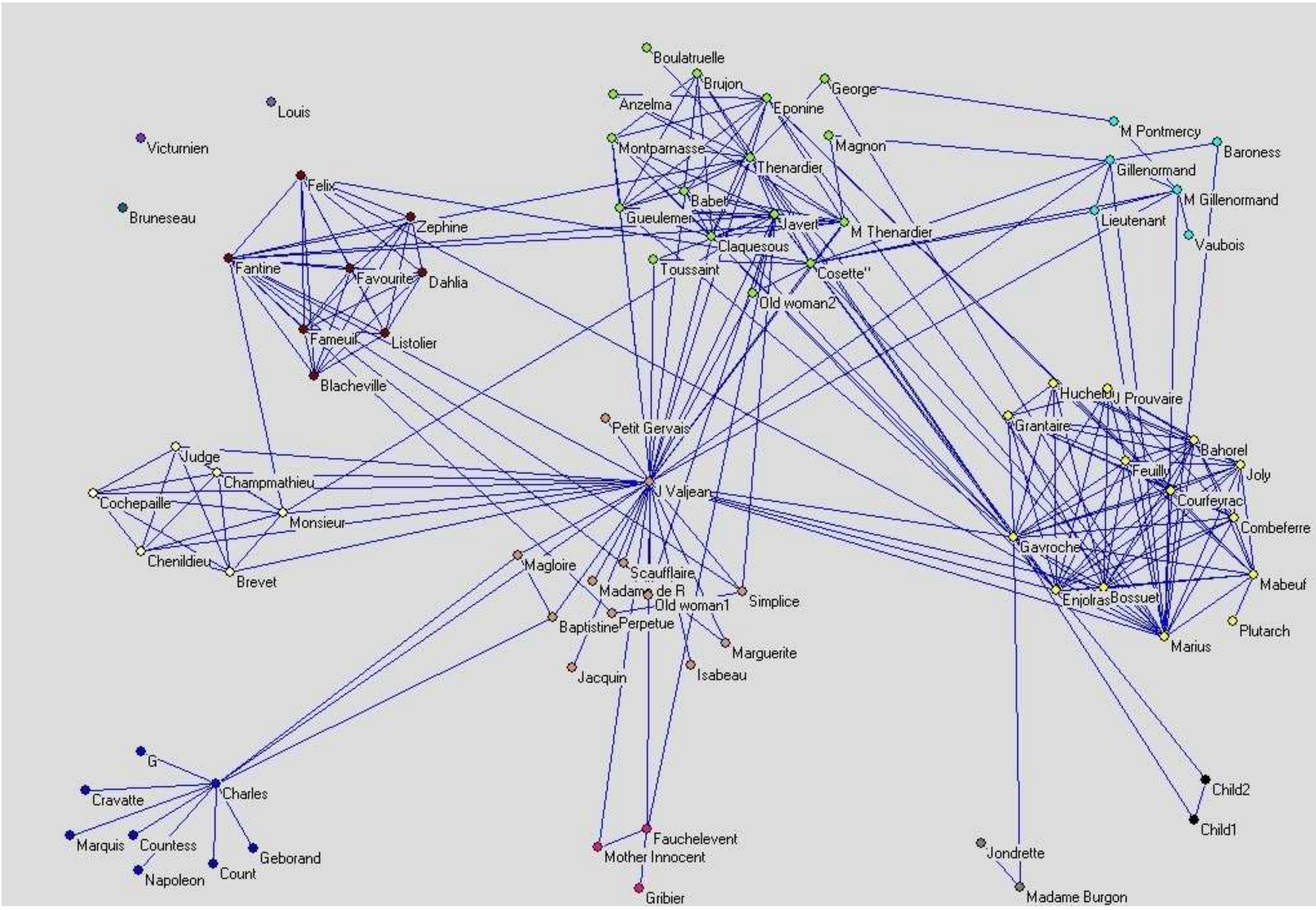


Figure 6.7: Grouping among the characters of Victor Hugo's *Les Miserables.* A total of 10 groups were recognized by the algorithm excluding the three groups that only contain one character each, which form the connected components of the network

2 were incorrectly classified, 6 were assigned to class 1 and 3 to class 2. On the other hand, normalized cut algorithm was unable to correctly classify 36 members of class 2 and 1 member of class 3, thus providing the classification accuracy of 79.21% as compared to 94.94% accuracy of Maximum Satisfactory Minimum Cut (MSMC) algorithm. We also

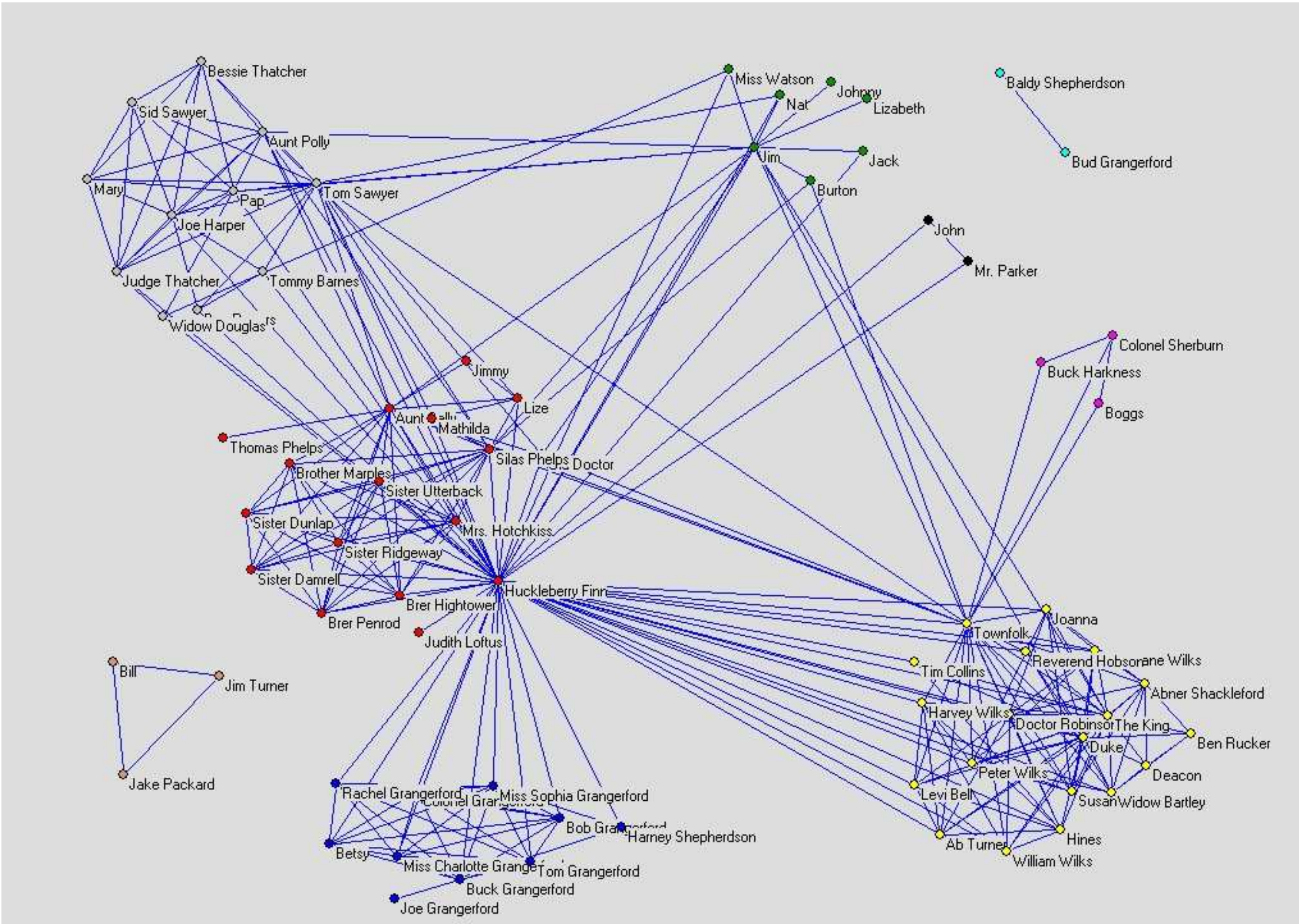


Figure 6.8: Nine groups that were found by the proposed algorithm among the characters of Mark Twain's Huckleberry Finn

tested the algorithm by using the data points of only two of the classes at a time. The algorithm MSMC algorithm classified 97.69% of data points belonging to class 1 and 2. The classification accuracy values between class 2 and class 3 and between class 1 and class 3 were 97.48% and 100% respectively. For the same data, the classification accuracy values for

Table 6.8: General information about Protein Localization Sites (Ecoli) data set

Data Set: Protein Localization Sites (Ecoli)	
Number of Instances	336
Number of Attributes	7
Class Distribution	C1 (cytoplasm): 143 instances C2 (inner membrane without signal sequence): 77 instances C3 (perisplasm): 52 instances C4 (inner membrane, uncleavable signal sequence): 35 instances C5 (outer membrane): 20 instances C6 (outer membrane lipoprotein): 5 instances C7 (inner membrane lipoprotein): 2 instances C8 (inner membrane, cleavable signal sequence) : 2 instances

normalized cut algorithm were 97.69%, 95.8% and 100%. The results of these experiments as well as the experiments on the other four data sets are summarized in Table 6.4.4.

The second data set, Firsher's *Iris Plant* database (Table 6.5) is composed of the measurements (sepal length/width and petal length/width) of 150 three different types of iris plants, 50 of each type. On the complete set of data, the MSMC algorithm correctly classified all 50 members of the first type. However, 9 members of second class and 11 members of third class were assigned the wrong class, which resulted in the classification accuracy of 86.66%. Normalized cut algorithm did better for this data set and correctly classified 50 members of the first type, 43 of the second and 45 of the third, a classification accuracy of 92%. As in the case of wine data set, we also tested the algorithms for each pair of classes. Both algorithms had classification accuracies of 100% while separating the members of first class from the other two. The classification accuracy of MSMC algorithm between class 2 and class 3 was 80% compared to 88% of normalized cut algorithm.

Similar experiments were performed for other three data sets. Other than the iris plant data set, MSMC algorithm consistently provided better accuracy than normalized cut. The results are presented in Table 6.4.4.

6.5 Conclusion

Clustering is still a developing field and is far from solved. The models and algorithms are neither general enough to be applicable to a variety of problems nor is there much consensus of which models to be used in which problems and why. This is because of the arbitrariness of the choice of models and the difficulty of independent interpretation of them outside the given applications. In addition, most of the models lead to NP-Hard problems and that requires a search for efficient approximate algorithms preferably with lower bounds on errors. Alliance is an intuitive model for clusters, groups or communities in a network. Using this model, we defined an objective function that is maximized by the optimal grouping of the vertices (in terms that the within group similarities and intergroup dissimilarities are maximized). We also presented an approximate algorithm to find such a grouping. We showed by using real world data that the algorithm performs well and is also comparable to other competing algorithms.

Table 6.9: Comparison of MSMC Algorithm and Normalized Cut Algorithm

Data Set	**Input Domain**	**Classification Performance**	
		MSMC Algorithm	Normalized Cut Algorithm
Wine Recognition	Class 1: 59 instants Class 2: 71 instants Class 3: 48 instants	Class 1: 59/59 Class 2: 62/71 Class 3: 48/48	Class 1: 59/59 Class 2: 35/71 Class 3: 47/48
		Accuracy: 94.94%	Accuracy: 79.21%
	Class 1: 59 instants Class 2: 71 instants	Class 1: 59/59 Class 2: 68/71	Class 1: 58/59 Class 2: 69/71
		Accuracy: 97.69%	Accuracy: 97.69%
	Class 1: 59 instants Class 3: 48 instants	Class 1: 59/59 Class 3: 48/48	Class 1: 59/59 Class 3: 48/48
		Accuracy: 100%	Accuracy: 100%
	Class 2: 71 instants Class 3: 48 instants	Class 2: 68/71 Class 3: 48/48	Class 2: 66/71 Class 3: 48/48
		Accuracy: 97.48%	Accuracy: 95.8%
Iris Plant	Class 1: 50 instants Class 2: 50 instants Class 3: 50 instants	Class 1: 50/50 Class 2: 41/50 Class 3: 39/50	Class 1: 50/50 Class 2: 43/50 Class 3: 45/50
		Accuracy: 86.66%	Accuracy: 92%
	Class 1: 50 instants Class 2: 50 instants	Class 1: 50/50 Class 2: 50/50	Class 1: 50/50 Class 2: 50/50
		Accuracy: 100%	Accuracy: 100%
	Class 1: 50 instants Class 3: 50 instants	Class 1: 50/50 Class 3: 50/50	Class 1: 50/50 Class 3: 50/50
		Accuracy: 100%	Accuracy: 100%
	Class 2: 50 instants Class 3: 50 instants	Class 2: 41/50 Class 3: 39/50	Class 2: 43/50 Class 3: 45/50
		Accuracy: 80%	Accuracy: 88%

Data Set	**Input Domain**	**Classification Performance**	
		MSMC Algorithm	Normalized Cut Algorithm
Hepatitis	Class 1: 32 instants Class 2: 123 instants	Class 1: 30/32 Class 2: 65/123	Class 1: 30/32 Class 2: 55/123
		Accuracy: 61.29%	Accuracy: 54.84%
Dermatology	Class 1: 112 instants Class 4: 49 instants	Class 1: 112/112 Class 4: 49/49	Class 1: 112/112 Class 4: 49/49
		Accuracy: 100%	Accuracy: 100%
	Class 1: 112 instants Class 6: 20 instants	Class 1: 112/112 Class 6: 20/20	Class 1: 106/112 Class 6: 20/20
		Accuracy: 100%	Accuracy: 95.45%
	Class 2: 61 instants Class 3: 72 instants	Class 2: 61/61 Class 3: 72/72	Class 2: 61/61 Class 3: 72/72
		Accuracy: 100%	Accuracy: 100%
	Class 2: 61 instants Class 5: 52 instants	Class 2: 61/61 Class 5: 52/52	Class 2: 61/61 Class 3: 52/52
		Accuracy: 100%	Accuracy: 100%
Protein Localization	Class 2: 77 instants Class 4: 52 instants	Class 2: 75/77 Class 4: 48/52	Class 2: 70/77 Class 4: 50/52
Sites (Ecoli)		Accuracy: 95.35%	Accuracy: 93.02%

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